

# Variable Mass Systems: The Rocket Equation

In this lecture, we consider the problem in which the mass of the body changes during the motion, that is,  $m$  is a function of  $t$ , i.e.  $m(t)$ . Although there are many cases for which this particular model is applicable, one of obvious importance to us are rockets. We shall see that a significant fraction of the mass of a rocket is the fuel, which is expelled during flight at a high velocity and thus, provides the propulsive force for the rocket.

As a simple model for this process, consider the cases sketched in a) through f) of the figure. In a) we consider 2 children standing on a stationary flat car. At  $t = 0$ , they both jump off with velocity **relative to the flat car** of  $u$ . (While this may not be a good model for children jumping off a flatcar, it is a good model for the expulsion of mass from a rocket where the mass flow from the choked nozzle occurs at a relative velocity  $u$  from the moving rocket.) In b) we consider that they jump off in sequence, each jumping with a velocity  $u$  relative to the then velocity of the flatcar, which will be different for the 2nd jumper since the car has begun to move as a result of jumper 1. For a), we have a final velocity of the flat car as

$$V_{2R} = \frac{2mu}{M + 2m} \quad (1)$$

for b) we have

$$V_{22} = \frac{mu}{M + 2m} + \frac{mu}{M + m} \quad (2)$$

where  $M$  is the mass of the flatcar, and  $m$  is the mass of the jumper. We can see the case b) gives a higher final velocity. We now consider cases with more blocks; we introduce the notation  $V_{NN}$  to be the final velocity with  $N$  blocks coming off one at a time. We use the notation  $V_{NR}$  to denote the reference velocity if the  $N$  blocks come off together. If we consider 3 blocks, we can see by inspection that

$$V_{3R} = \frac{3mu}{M + 3m} \quad (3)$$

and

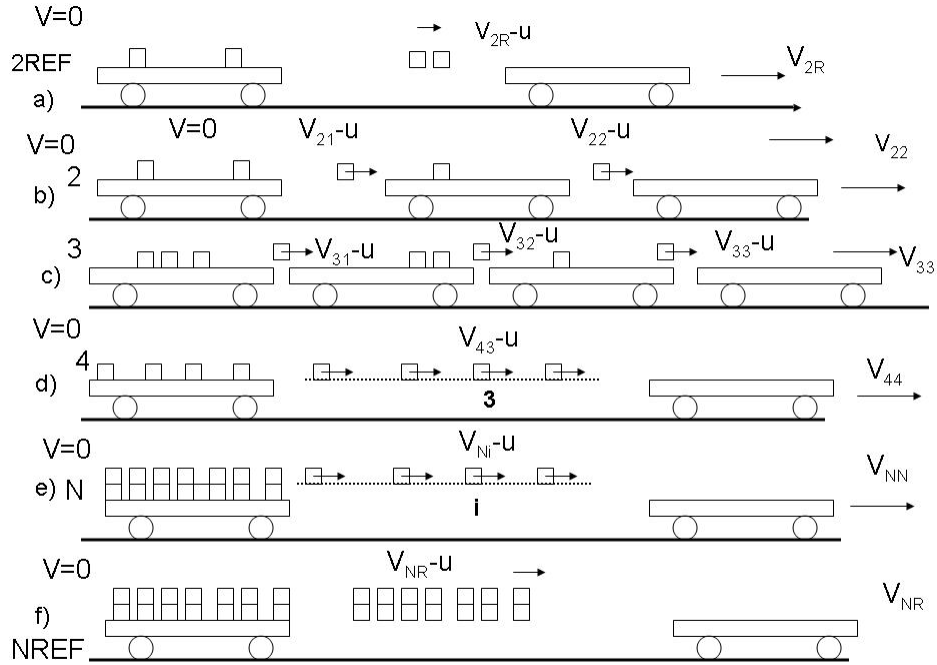
$$V_{33} = \frac{mu}{M + 3m} + \frac{mu}{M + 2m} + \frac{mu}{M + m} \quad (4)$$

While for 4 blocks

$$V_{4R} = \frac{4mu}{M + 4m} \quad (5)$$

and

$$V_{44} = \frac{mu}{M + 4m} + \frac{mu}{M + 3m} + \frac{mu}{M + 2m} + \frac{mu}{M + m} \quad (6)$$

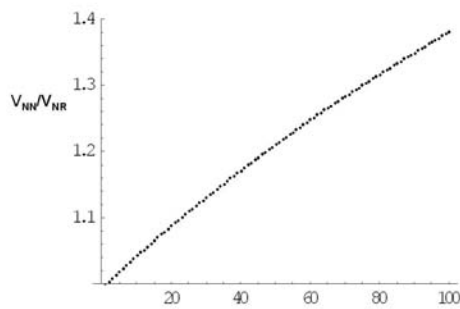


Generalizing this to arbitrary N, we can write

$$V_{NR} = \frac{Nmu}{M + Nm} \quad (7)$$

and

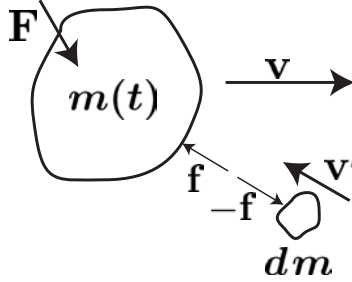
$$V_{NN} = \sum_{i=1}^N \frac{mu}{M + im} \quad (8)$$



The figure shows the results for  $V_{100,100}/V_{100R}$ , the increase in velocity as each block comes off. We choose 100 blocks with a mass  $m$  equal to  $.01 M$ ; therefore the 100 blocks with  $m = .01$  equal the mass  $M$ , the propellant ratio of the "rocket" is 50%. The final velocity of the cart is 40% higher than would be obtained if the masses come off together. Momentum is conserved. The sum of the momentum of the cart plus all the individual mass points is equal to zero. However, the momentum of all the particles left behind is of very little interest to us. We care only about the final velocity of the cart.

To analyze this question we must consider a system of variable mass, and the process by which it gains velocity as a result of ejecting mass. While our previous discussion provided a discrete model of this process, we now consider a continuous process, obtaining a relation for  $v(t)$  as a function of the momentum interaction between the system and the external world.

We shall start by considering a body with velocity  $\mathbf{v}$  and external forces  $\mathbf{F}$ , gaining mass at a rate  $\dot{m} = dm/dt$ . Let us look at the process of gaining a small amount of mass  $dm$ . Let  $\mathbf{v}'$  be the velocity of  $dm$  before it is captured by  $m$ , and let  $\mathbf{f}$  represent the average value of the impulsive forces that  $dm$  exerts on  $m$  during the short interval  $dt$ , in which the capturing takes place. By Newton's third law,  $dm$  will experience a force  $-\mathbf{f}$ , exerted by  $m$ , over the same  $dt$ .



We can now examine the capture process from the point of view of  $dm$  and equate the impulse,  $-\mathbf{f}dt$ , to the change in linear momentum of  $dm$ ,

$$-\mathbf{f}dt = dm(\mathbf{v} + d\mathbf{v} - \mathbf{v}'). \quad (9)$$

Here,  $\mathbf{v} + d\mathbf{v}$  is the velocity of  $m$  (and  $dm$ ) after impact. Analogously, from the point of view of  $m$ , we write

$$\mathbf{F}dt + \mathbf{f}dt = m(\mathbf{v} + d\mathbf{v}) - m\mathbf{v} = m d\mathbf{v}. \quad (10)$$

The term  $dm d\mathbf{v}$  in equation (9) is a higher order term and will disappear when we take limits. The impulse due to the contact force can be eliminated by combining equations (9) and (10),

$$\mathbf{F}dt - dm(\mathbf{v} - \mathbf{v}') = m d\mathbf{v},$$

or, dividing through by  $dt$ ,

$$m \frac{d\mathbf{v}}{dt} = \mathbf{F} - (\mathbf{v} - \mathbf{v}') \frac{dm}{dt} = \mathbf{F} + (\mathbf{v}' - \mathbf{v}) \frac{dm}{dt}. \quad (11)$$

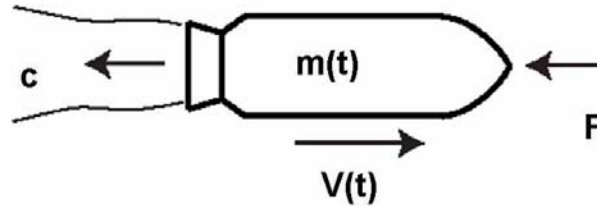
Here,  $\mathbf{v}' - \mathbf{v}$  is the velocity of  $dm$  relative to  $m$ . This expression is valid when  $dm/dt > 0$  (mass gain) and when  $dm/dt < 0$  (mass loss). If we compare this expression to the more familiar form of Newton's law for a particle of fixed mass  $m \frac{d\mathbf{v}}{dt} = \mathbf{F}$ , we see that the term  $(\mathbf{v}' - \mathbf{v})dm/dt$  is an additional force on  $m$  which is due to the gain (or loss) of mass. Equation (11) can also be written as

$$\frac{d(m\mathbf{v})}{dt} = \mathbf{F} + \mathbf{v}' \frac{dm}{dt},$$

where  $v'$  is the velocity of the captured (or expelled) mass relative to the velocity of the mass  $m$ . This shows that, for systems involving variable mass, the usual expression stating conservation of linear momentum,  $d(m\mathbf{v})/dt = \mathbf{F}$ , is only applicable when the initial (final) velocity of the captured (expelled) mass,  $\mathbf{v}'$ , is zero. The behavior of  $m(t)$  is not an unknown, but is specified according to the characteristics of the rocket. In most cases,  $\frac{dm}{dt}$  is a constant and negative. In some cases, the behavior of  $m(t)$  may be determined by a control system. In any case, it is a given quantity.

## The Rocket Equation

We consider a rocket of mass  $m$ , moving at velocity  $\mathbf{v}$  and subject to external forces  $\mathbf{F}$  (typically gravity and drag). The rocket mass changes at a rate  $\dot{m} = dm/dt$ , with a velocity vector  $\mathbf{c}$  relative to the rocket. We shall assume that the magnitude of  $\mathbf{c}$  is constant.



The velocity of the gas observed from a stationary frame will be  $\mathbf{v}' = \mathbf{v} + \mathbf{c}$ . In this frame,  $\mathbf{c}$  is a vector aligned along the flight path in a negative direction.  $\mathbf{c} = -c\mathbf{t}$ , where  $\mathbf{t}$  is the unit direction along the flight path. Thus,

$$m \frac{d\mathbf{v}}{dt} = \mathbf{F} + \mathbf{c} \frac{dm}{dt}. \quad (12)$$

The term  $\mathbf{T} = \mathbf{c} dm/dt$  is called the *thrust* of the rocket and can be interpreted as an additional force on the rocket due to the gas expulsion.

Equation (12) is a vector equation which can be projected along the direction of  $\mathbf{v}$  (tangent to the path). Thus,

$$m \frac{dv}{dt} = F_t - c \frac{dm}{dt} = F_t + T, \quad (13)$$

where  $F_t$  is the tangential component of  $\mathbf{F}$ ,  $v$  and  $c$  are the magnitudes of  $\mathbf{v}$  and  $\mathbf{c}$  respectively, and we have assumed that  $\mathbf{c}$  is parallel and has opposite direction to  $\mathbf{v}$ . The magnitude of the thrust is  $T = -c\dot{m}$ . Note that for a rocket,  $\dot{m}$  will be negative (mass is lost). If the force  $F_t$  is known, this equation can be integrated in time to yield an expression for the velocity as a function of time.

Let us consider some simple cases:

**No External Forces:**  $F_t = 0$

If gravity and drag effects are neglected, we have,

$$m \frac{dv}{dt} = -c \frac{dm}{dt}$$

or, integrating between an initial time  $t_0$ , and a final time  $t$ ,

$$\Delta v = v - v_0 = -c(\ln m_f - \ln m_0) = -c \ln \frac{m}{m_0} = c \ln \frac{m_0}{m_f}. \quad (14)$$

Alternatively, this expression can be cast as the well known rocket equation,

$$m = m_0 e^{-\Delta v/c}, \quad (15)$$

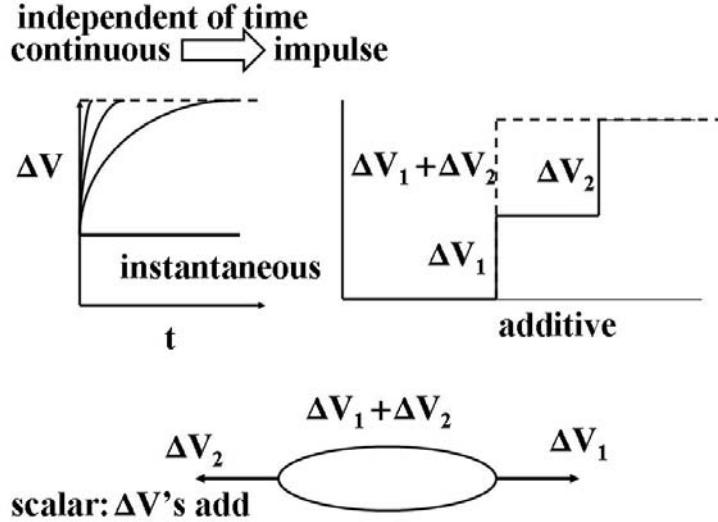
which gives the mass of the rocket at a time  $t$ , as a function of the initial mass  $m_0$ ,  $\Delta v$ , and  $c$ . The mass of the propellant,  $m_{propellant}$ , is given by,

$$m_{propellant} = m_0 - m = m_0(1 - e^{-\Delta v/c}).$$

From the above equations, we see that for a given  $\Delta v$  and  $m_0$ , increasing  $c$  increases  $m$  (payload plus structure) and decreases  $m_{propellant}$ . Unfortunately, we can only choose  $c$  as high as the current technology will allow. For current chemical rockets,  $c$  ranges from 2500-4500 m/sec. Ion engines can have  $c$ 's of roughly  $10^5$  m/sec. However, since relatively few particles are involved, the thrust is quite low, whereas chemical rockets have high mass flows.

Equation (13) has some quite remarkable properties, as sketched below. First, the  $\Delta V$  for a given  $m_0/m_f$  is independent of time. More strikingly, if we compare the final  $\Delta V$  to the flat car situation where all the mass was ejected instantaneously rather than continuously, we see a substantial gain in  $\Delta V$  from the continuous process. However, the final  $\Delta V$  for the continuous process is independent of the actual time span of the process, as shown in the sketch. Thus in the limit as the continuous process occurs quickly, we will obtain the result for the continuous process. We will refer to the rapid continuous process as a  $\Delta V$  impulse. When we consider rocket thrusting during orbital maneuvers, we will consider only impulsive thrust.

A second property, is that the final  $\Delta V$  from a series of firings is independent of the details and a function only of the total mass change over the processes. Consider a first thrust producing a  $\Delta V$ . Then  $\Delta V_1 = c \ln \frac{m_0}{m_1}$ . Now the mass is  $m_1$  and a second thrust produces a  $\Delta V_2 = c \ln \frac{m_1}{m_2}$ . By a property of logarithms, adding these two  $\Delta V$ 's together gives  $\Delta V_f = \Delta V_1 + \Delta V_2 = c \ln \frac{m_0}{m_1} + c \ln \frac{m_1}{m_2} = c \ln \frac{m_0}{m_2}$  where  $m_2$  is the final mass of the vehicle  $m_f$ .



Another property is that the relationship between thrust and  $\Delta V$  is a positive scalar, always additive. Whether we thrust in the positive or negative direction,–or to the side– it always consumes propellant, we never get anything back

Note that if  $\dot{m}$  is constant, we can write  $m(t) = m_0 + \dot{m}t$ , and substitute into equation 14 to obtain an expression for  $v$  as a function of  $t$ ,

$$v = v_0 - c \ln\left(1 + \frac{\dot{m}}{m_0} t\right) .$$

Recall that according to the convention used,  $\dot{m}$  is negative as the mass decreases with time.

**Gravity:**  $F_t = -mg$

A constant gravitational field acting in the opposite direction to the velocity vector can be easily incorporated. In this case, equation 13 becomes,

$$m \frac{dv}{dt} = -mg - c \frac{dm}{dt},$$

which can be integrated to give

$$v = v_0 - c \ln \frac{m}{m_0} - gt = v_0 - c \ln \frac{m}{m_0} - g \frac{m - m_0}{\dot{m}}. \tag{16}$$

This solution assumes that  $c\dot{m} > m_0g$  at  $t=0$ . If this is not true, the rocket will sit on the pad, burning fuel until the remaining mass satisfies this requirement.

**Gravity Plus Drag:**  $F_t = -mg - D$

The effect of the drag force,  $D$ , is harder to quantify. It turns out that for many important applications drag effects are very small. The drag force is characterized in terms of a drag coefficient,  $C_D$ . Thus,

$$D = \frac{1}{2} \rho v^2 A C_D ,$$

where  $A$  is the cross-sectional area of the rocket. The air density changes with altitude  $z$ , and may be approximated by

$$\rho = \rho_0 e^{-z/H} ,$$

where  $H \approx 8000$  m is the so-called “scale height” of the atmosphere, and  $\rho_0$  is the air density at sea level. It turns out that the differential equation that results for the velocity cannot be integrated explicitly and, in practice, needs to be integrated numerically. It is interesting to note, however, that the effect of drag losses is usually quite small, and it is often reasonable to ignore it in a first calculation. In order to see the importance of  $D$  versus the effect of gravity, we can estimate the value of the ratio  $D/mg$ . At conditions typical for maximum drag,  $\rho \approx 0.25$  kg/m<sup>3</sup> and  $v = 700$  m/s. Considering a rocket of 12,000 kg with a cross section of  $A = 1$  m<sup>2</sup> and  $C_D = 0.2$ , we have,

$$\frac{\rho A C_D v^2}{2mg} = \frac{0.25 \times 1 \times 0.2 \times 700^2}{2 \times 12,000 \times 9.8} = 0.021 ,$$

which indicates that the drag force is only about 2% of the gravity force.

**Note (Optional)**

**Gravity Losses [1]**

Let us consider a rocket providing a constant thrust  $T = -c\dot{m}$  which is launched vertically upwards from rest. Neglecting drag but considering gravity forces, the velocity is given by expression (16) which can be re-written as

$$v = c \left( \ln \frac{1}{\mu} - \frac{1-\mu}{n} \right) , \quad (17)$$

where we have introduced  $\mu = m/m_0$  and the thrust induced acceleration measured in “gees”,  $n = -c\dot{m}/(gm_0)$ . Now,  $v = dz/dt$ , and the above equation can be integrated to give

$$z = \frac{c^2}{gn} \left( 1 - \mu \ln \frac{1}{\mu} - \mu - \frac{(1-\mu)^2}{2n} \right) . \quad (18)$$

If we consider equations (17) and (18) at burnout time (when all the propellant is consumed), the mass fraction will be  $\mu = (m_0 - m_{propellant})/m_0$ . After this time, the rocket will coast in free-flight. The total energy per unit mass at this (and all subsequent times) is

$$E = \frac{v^2}{2} + gz = c^2 \left( \frac{1}{2} \left( \ln \frac{1}{\mu} \right)^2 + \frac{1-\mu - \ln(1/\mu)}{n} \right) .$$

Since,  $0 < \mu < 1$ ,  $1 - \mu - \ln(1/\mu)$  is negative and consequently, higher  $n$  means larger final energy per unit mass  $E$ , tending to the ideal limit

$$E_{ideal} = \frac{1}{2} \left( c \ln \frac{1}{\mu} \right)^2 ,$$

which is that obtained for an impulsive start. Reducing  $n$  means applying smaller thrust for a longer time, and, as the above equation shows there is a price to be paid in the eventual energy of the payload. One way to quantify this loss is to define an equivalent impulsive velocity  $(v_0)_{eq}$  such as to achieve the energy per

unit mass  $E$  in a very short time, hence, at ground level,

$$(v_0)_{eq} = c \sqrt{\left(\ln \frac{1}{\mu}\right)^2 + 2 \frac{1 - \mu - \ln(1/\mu)}{n}},$$

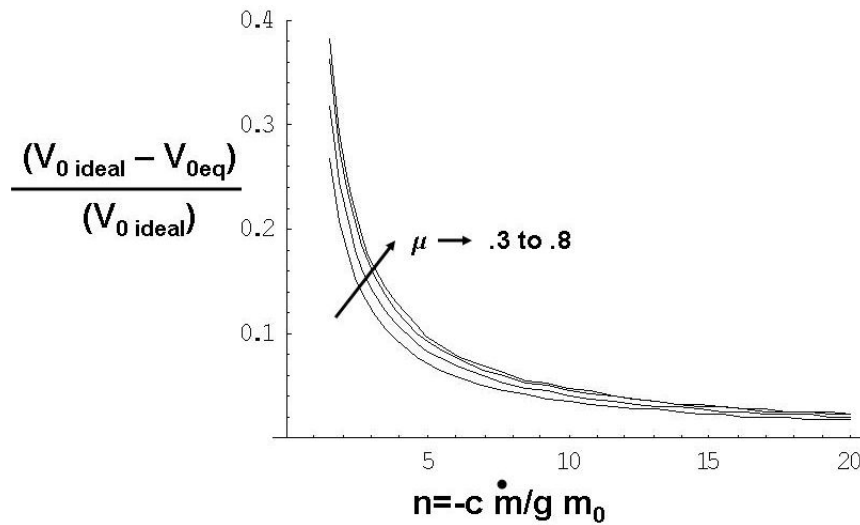
and to compare this to the ideal impulsive velocity

$$(v_0)_{ideal} = c \ln \frac{1}{\mu}.$$

Some representative fractional losses  $((v_0)_{ideal} - (v_0)_{eq}) / (v_0)_{ideal}$ , are shown below,

	$\mu = 0.7$	$\mu = 0.5$	$\mu = 0.3$
$n = 1.5$	0.3628	0.3188	0.2676
$n = 3$	0.1615	0.1444	0.1235
$n = 10$	0.0456	0.0410	0.0354

The table shows losses of  $\sim 15\%$  for  $n = 3$ ,  $\sim 4\%$  for  $n = 10$ . (Recall that  $n$  must be greater than 1 for immediate lift off from the launch pad.) The losses are severe for  $n$  as low as 1.5. The figure shows a graph of these losses as a function on  $n$  for  $\mu = .8, .7, .5$ , and  $.3$ .



### Example

### Single vs. Two Stage Rockets [1]

#### Single stage

To achieve an orbital speed of  $v = 7600$  m/s, we require an *ideal*  $\Delta v$  of about 9000 m/s where the extra velocity is needed to overcome gravity and drag. Chemical rockets produce exhaust jets at velocities of  $c \approx 2500 - 4500$  m/s. Using the higher  $c$ , if we wish to place a payload in orbit with a single stage rocket, we have a mass ratio (mass at burn out,  $m_f$ , divided by initial mass,  $m_0$ ) of

$$\frac{m_f}{m_0} = e^{-9000/4500} = e^{-2} = 0.135.$$



The mass  $m_f$  must include all components of the rocket infrastructure, including the engine, empty tank, guidance equipment, etc., as well as the payload. The mass of the propellant will be  $m_0 - m_f = (1 - 0.135)m_0 = 0.865m_0$ . If we assume that the tank plus engine are the main contributors to the weight of the rocket and weigh 10% of the propellant,  $0.087m_0$ , we have,

$$m_{payload}^{1stage} = (0.135 - 0.087)m_0 = 0.048m_0 .$$

Clearly, there is not much margin here, and, in fact, single-stage-to-orbit vehicles have yet to be engineered successfully. The alternative solution is to subdivide the rocket, so that empty tanks are dropped when they are no longer needed.

### Two stages

Consider now a two stage vehicle. We shall again assume that each empty tank plus its engine weighs 10% of the propellant it carries. The required  $\Delta v$  is now subdivided into two  $\Delta v$ 's of 4500 m/s each.

#### First stage

If  $m_0$  is the initial mass and  $m_1$  is the mass after burn out, we have,

$$m_1 = e^{-4500/4500}m_0 = e^{-1}m_0 = 0.368m_0$$

The fuel burnt will be  $m_0 - m_1 = (1 - 0.368)m_0 = 0.632m_0$ , and the weight of the tank and engine to be dropped will be  $0.0632m_0$ , leaving an initial mass for the second stage,  $m_2$ , of

$$m_2 = 0.368m_0 - 0.063m_0 = 0.295m_0 .$$

#### Second stage

After burn out,

$$m_f = e^{-1}m_2 = e^{-1}0.295m_0 = 0.109m_0.$$

The fuel burnt in this stage will be  $m_2 - m_f = (0.295 - 0.109)m_0 = 0.186m_0$  and the weight of the tank plus engine will be  $0.019m_0$ , leaving for the payload

$$m_{payload}^{2stage} = (0.109 - 0.019)m_0 = 0.090m_0,$$

which is still very small but about *twice* the size of the payload obtained for the single stage rocket.

### Note

### Launch Strategy

The earth rotates towards the East with a period of about 23.9 hours. This results in a tangential velocity of  $v_t = 466m/sec.$  at the equator. The corresponding value for higher latitudes is  $v_t = 466m/sec * Cos\theta_0$ , where  $\theta_0$  is the latitude of position on the earth. In most situations, we desire to use this tangential velocity to add to the energy available to place a satellite in orbit. Therefore, most launch sites are located as close

to the equator as possible and the launch occurs with a velocity component directed towards the east. The table shows the latitude of the current major international launch sites. Of interest is "Sea Launch", which leaves the US west coast with a sea-going launch platform and control ship and launches very close to the equator. Other criterion for launch site is safety; a failed launch should not put populations in danger.

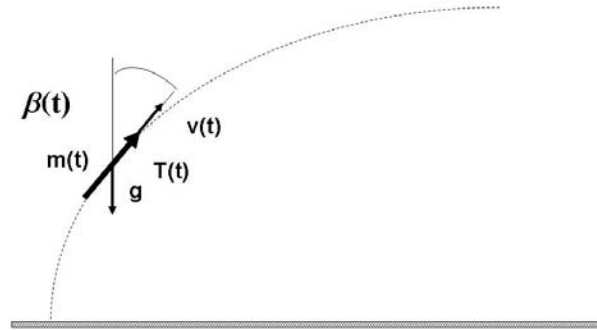
Kennedy	28.5°	Tanegashima	30.4
Vandenburg	34.4°	Xichang	28.25
Baikonur	45.6°	Kourou	5.2
Sea Launch	0°19 min		

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**Note**

**Gravity Turn**

The model we have introduced for rocket launch is incomplete if our goal is to place a spacecraft into orbit. Orbital launch requires that the flight end with a roughly horizontal velocity at orbital speed. One useful maneuver to accomplish this transition is called the gravity turn. In this maneuver, gravity acts to turn the trajectory of the rocket towards the horizontal.



This maneuver offers two main advantages over a conventional thrust-controlled trajectory where the rocket's own thrust is used to steer the vehicle. First, any thrust used in changing the rocket's direction is not being used to accelerate the vehicle into orbit, constituting a loss which can be reduced by using gravity to steer the vehicle onto its desired trajectory. Second, and more importantly, because the force of gravity is doing the steering during the initial ascent phase of the launch the ship can maintain low or even zero angle of attack. This minimizes transverse stress on the launch vehicle; allowing for a weaker, and thus lighter, launch vehicle. For a constant thrust, and a variable mass, the details of the trajectory and motion require a numerical solution. Using local intrinsic coordinate, the governing equations for the gravity turn are

$$\dot{v} = T/m(t) - g \cos \beta \quad (19)$$

$$v\dot{\beta} = g \sin \beta \quad (20)$$

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**Note****Specific Impulse**

The specific impulse  $I_{sp}$ , with units of seconds, is often used in practice to characterize the performance of a rocket engine. The definition of the specific impulse is the magnitude of the thrust divided by the propellant weight flow rate,

$$I_{sp} = \frac{T}{\dot{m}g} = \frac{c}{g}.$$

Typical values of  $I_{sp}$  are around 300 s for solid propellants and up to 500 s for higher energy fuels.

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**ADDITIONAL READING**

J.L. Meriam and L.G. Kraige, *Engineering Mechanics, DYNAMICS*, 5th Edition

4/6, 4/7

**References**

- [1] M. Martinez-Sanchez, *Unified Engineering Notes*, Course 95-96.

source:<http://ocw.mit.edu/courses/aeronautics-and-astronautics/16-07-dynamics-fall-2009/lecture-notes/>

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