Surface and Contact Stress

The concept of the force is fundamental to mechanics and many important problems can be cast in terms of forces only, for example the problems considered in Chapter 2. However, more sophisticated problems require that the action of forces be described in terms of *stress*, that is, force divided by area. For example, if one hangs an object from a rope, it is not the weight of the object which determines whether the rope will break, but the weight divided by the cross-sectional area of the rope, a fact noted by Galileo in 1638.

3.1.1 Stress Distributions

As an introduction to the idea of stress, consider the situation shown in Fig. 3.1.1a: a block of mass m and cross sectional area A sits on a bench. Following the methodology of Chapter 2, an analysis of a free-body of the block shows that a force equal to the weight mg acts upward on the block, Fig. 3.1.1b. Allowing for more detail now, this force will actually be distributed over the surface of the block, as indicated in Fig. 3.1.1c. Defining the stress to be force divided by area, the stress acting on the block is

$$\sigma = \frac{mg}{A} \tag{3.1.1}$$

The unit of stress is the Pascal (Pa): 1Pa is equivalent to a force of 1 Newton acting over an area of 1 metre squared. Typical units used in engineering applications are the kilopascal, kPa (10^3 Pa), the megapascal, MPa (10^6 Pa) and the gigapascal, GPa (10^9 Pa).

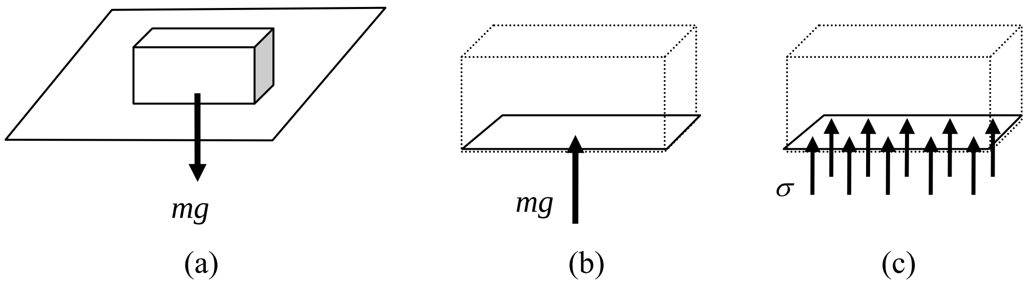

Figure 3.1.1: a block resting on a bench; (a) weight of the block, (b) reaction of the bench on the block, (c) stress distribution acting on the block

The stress distribution of Fig. 3.1.1c acts on the block. By Newton's third law, an equal and opposite stress distribution is exerted by the block on the bench; one says that the weight force of the block is *transmitted* to the underlying bench.

The stress distribution of Fig. 3.1.1 is **uniform**, i.e. constant everywhere over the surface. In more complex and interesting situations in which materials contact, one is more likely to obtain a *non-uniform* distribution of stress. For example, consider the case of a metal ball being pushed into a similarly stiff object by a force F, as illustrated in Fig. 3.1.2.[1] Again, an equal force F acts on the underside of the ball, Fig. 3.1.2b. As with the block, the force will actually be distributed over a **contact region**. It will be shown in Part II that the ball (and the large object) will deform and a circular contact region will arise where the ball and object meet[2], and that the stress is largest at the centre of the contact surface, dying away to zero at the edges of contact, Fig. 3.1.2c ($\sigma_1 > \sigma_2$ in Fig. 3.1.2c). In this case, with stress σ not constant, one can only write, Fig. 3.1.2d,

$$F = \int_A dF = \int_A \sigma dA \tag{3.1.2}$$

The stress varies from point to point over the surface but the sum (or integral) of the stresses (times areas) equals the total force applied to the ball.

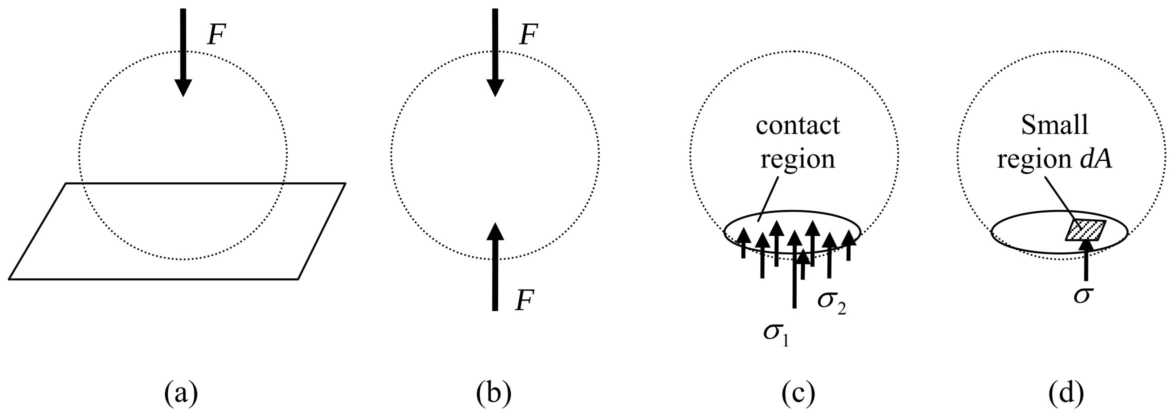

Figure 3.1.2: a ball being forced into a large object, (a) force applied to ball, (b) reaction of object on ball, (c) a non-uniform stress distribution over the contacting surface, (d) the stress acting on a small (infinitesimal) area

A given stress distribution gives rise to a resultant force, which is obtained by integration, Eqn. 3.1.2. It will also give rise to a resultant moment. This is examined in the following example.

Example

Consider the surface shown in Fig. 3.1.3, of length 2m and depth 2m (into the page). The stress over the surface is given by $\sigma = x$ kPa, with x measured in m from the left-hand side of the surface.

The force acting on an element of length dx at position x is (see Fig. 3.1.3b)

$$dF = \sigma\, dA = \left(x\,\text{kPa}\right) \times \left(dx\,\text{m} \times 2\,\text{m}\right)$$

[1] the weight of the ball is neglected here
[2] the radius of which depends on the force applied and the materials in contact

The resultant force is then, from Eqn. 3.1.2

$$F = \int_A dF = 2\int_0^2 x dx\left(\text{kPa m}^2\right) = 4\,\text{kN}$$

The moment of the stress distribution is given by

$$M_0 = \int_A dM = \int_A \sigma \times l\, dA \tag{3.1.3}$$

where l is the length of the moment-arm from the chosen axis.

Taking the axis to be at $x = 0$, the moment-arm is $l = x$, Fig. 3.1.3b, and

$$M_{x=0} = \int_A dM = 2\int_0^2 x \times x\, dx\left(\text{kPa m}^3\right) = \frac{16}{3}\,\text{kN m}$$

Taking moments about the right-hand end, $x = 2$, one has

$$M_{x=2} = \int_A dM = -2\int_0^2 x \times (2 - x) dx\left(\text{kPa m}^3\right) = -\frac{8}{3}\,\text{kN m}$$

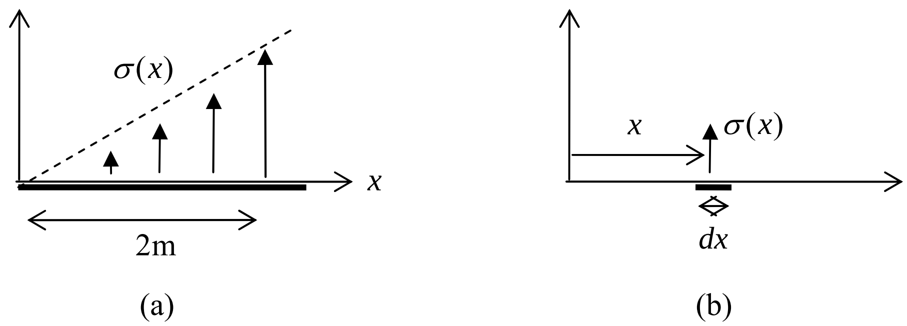

Figure 3.1.3: a non-uniform stress acting over a surface; (a) the stress distribution, (b) stress acting on an element of size d*x*

■

3.1.2 Equivalent Forces and Moments

Sometimes it is useful to replace a stress distribution σ with an **equivalent force** F, i.e. a force equal to the resultant force of the distribution and one which also give the same moment about any axis as the distribution. Formulae for equivalent forces are derived in what follows for triangular and arbitrary linear stress distributions.

Triangular Stress Distribution

Consider the triangular stress distribution shown in Fig. 3.1.4. The stress at the end is σ_0, the length of the distribution is L and the thickness "into the page" is t. The equivalent force is, from Eqn. 3.1.2,

$$F = t\sigma_0 \int_0^L \frac{x}{L} dx = \frac{1}{2}\sigma_0 Lt \tag{3.1.4}$$

which is just the average stress times area. The point of action of this force should be such that the moment of the force is equivalent to the moment of the stress distribution. Taking moments about the left hand end, for the distribution one has, from 3.1.3,

$$M_o = t\int_0^L x\sigma(x)dx = \frac{1}{3}\sigma_0 L^2 t$$

Placing the force at position $x = x_c$, Fig. 3.1.4, the moment of the force is $M_o = (\sigma_0 Lt/2)x_c$. Equating these expressions leads to the position at which the equivalent force acts:

$$x_c = \frac{2}{3}L. \tag{3.1.5}$$

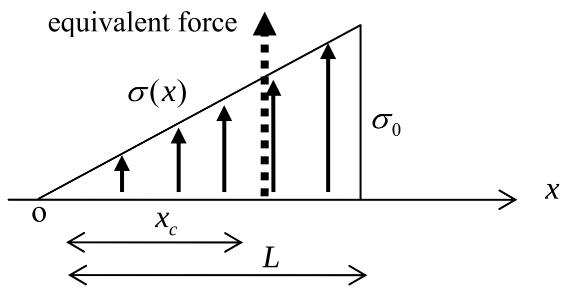

Figure 3.1.4: triangular stress distribution and equivalent force

Note that the moment about *any* axis is now the same for both the stress distribution and the equivalent force.

■

Arbitrary Linear Stress Distribution

Consider the linear stress distribution shown in Fig. 3.1.5. The stress at the ends are σ_1 and σ_2 and this time the equivalent force is

$$F = t\int_0^L [\sigma_1 + (\sigma_2 - \sigma_1)(x/L)]dx = Lt(\sigma_1 + \sigma_2)/2 \tag{3.1.6}$$

Taking moments about the left hand end, for the distribution one has

$$M_o = t\int_0^L x\sigma(x)dx = L^2 t(\sigma_1 + 2\sigma_2)/6$$

The moment of the force is $M_o = Lt(\sigma_1 + \sigma_2)x_c / 2$. Equating these expressions leads to

$$x_c = \frac{L(\sigma_1 + 2\sigma_2)}{3(\sigma_1 + \sigma_2)} \tag{3.1.7}$$

Eqn. 3.1.5 follows from 3.1.7 by setting $\sigma_1 = 0$.

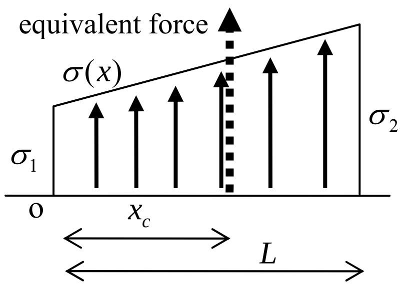

Figure 3.1.5: a non-uniform stress distribution and equivalent force

■

The Centroid

Generalising the above cases, the line of action of the equivalent force for any arbitrary stress distribution $\sigma(x)$ is

$$\boxed{x_c = \frac{t\int x\sigma(x)dx}{t\int \sigma(x)dx} = \frac{\int xdF}{F}} \qquad \textbf{Centroid} \tag{3.1.8}$$

This location is known as the **centroid** of the distribution.

Note that most of the discussion above is for two-dimensional cases, i.e. the stress is assumed constant "into the page". Three dimensional problems can be tackled in the same way, only now one must integrate two-dimensionally over a surface rather than one-dimensionally over a line.

Also, the forces considered thus far are *normal* forces, where the force acts perpendicular to a surface, and they give rise to **normal stresses**. Normal stresses are also called **pressures** when they are compressive as in Figs. 3.1.1-2.

3.1.3 Shear Stress

Consider now the case of **shear forces**, that is, forces which act tangentially to surfaces.

A normal force F acts on the block of Fig. 3.1.6a. The block does not move and, to maintain equilibrium, the force is resisted by a friction force $F = \mu mg$, where μ is the coefficient of friction. A free body diagram of the block is shown in Fig. 3.1.6b. Assuming a uniform distribution of stress, the stress and resultant force arising on the surfaces of the block and underlying object are as shown. The stresses are in this case called **shear stresses**.

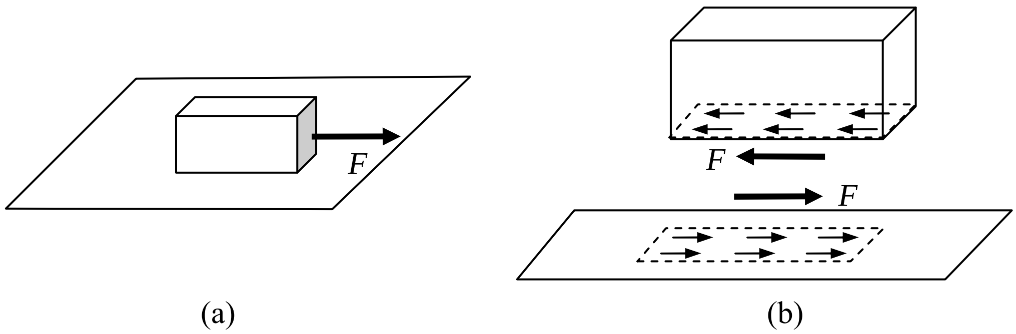

Figure 3.1.6: shear stress; (a) a force acting on a block, (b) shear stresses arising on the contacting surfaces

3.1.4 Combined Normal and Shear Stress

Forces acting inclined to a surface are most conveniently described by decomposing the force into components normal and tangential to the surface. Then one has both normal stress σ_N and shear stress σ_S, as in Fig. 3.1.7.

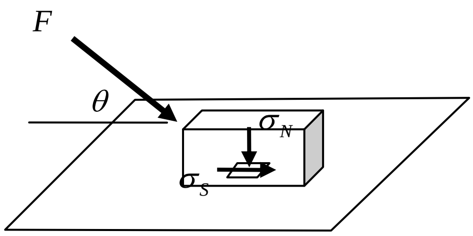

Figure 3.1.7: a force F giving rise to normal and shear stress over the contacting surfaces

The stresses considered in this section are examples of **surface stresses** or **contact stresses**. They arise when materials meet at a common surface. Other examples would be sea-water pressurising a material in deep water and the stress exerted by a train wheel on a train track.