Strain Transformation equations:

1.1 General state of strain

When a body is subjected to stress, it undergoes deformation. When deformation happens, the points in the body are subjected to displacement. Such displacements include deformation, translation and rotation. Normal stress causes deformation. Shear stress can cause both rotation and translation. While considering normal strain, we ignore rotation and translation.

Deformations in solids may also be due to volume changes or distortion, which is shape change. Displacements in rigid bodies are a linear function of distance.

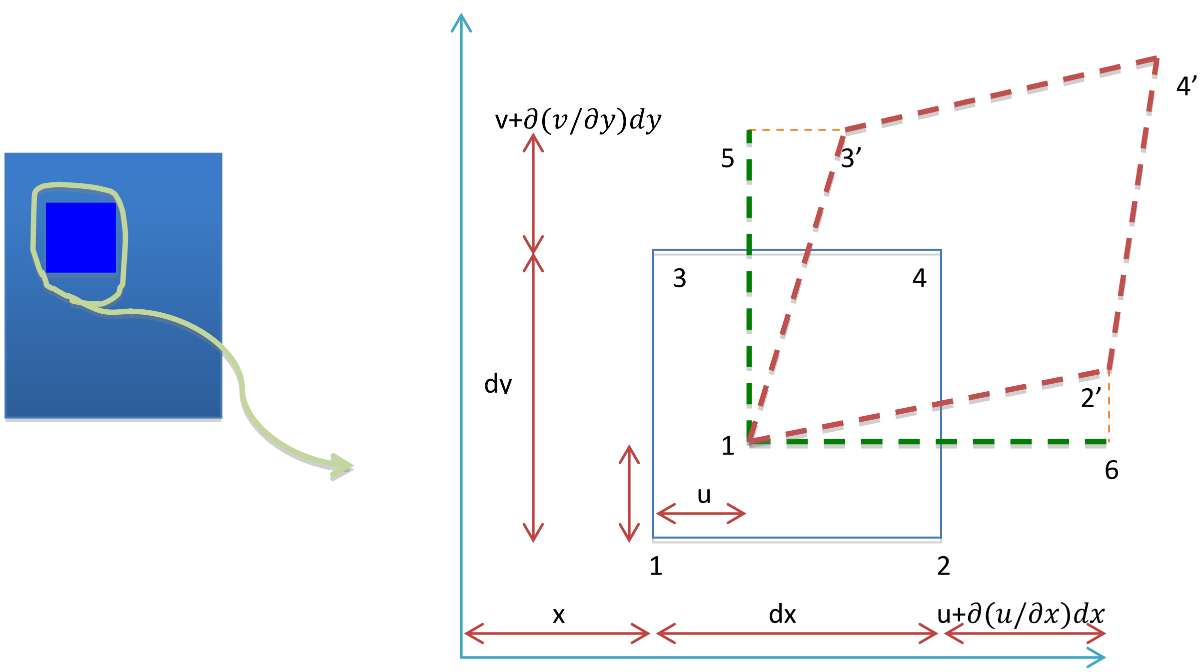

Fig.2.1.1: State of strains on a body – normal and shear

Consider a small elemental plane of a solid subjected to elastic deformation, as shown above. The sides of the element undergo distortion as shown by dotted lines. The side 1-2 gets translated and sheared to 1'-2', Let us ignore the rotation of the elements. The displacements of various points in the element is assumed to be linearly proportional to their distance. Farther points in the element will undergo more displacement. This assumption is valid for small displacements and elastic bodies.

Point 1 has a displacement of u along x axis and v along y axis.

Neglecting rotation, the side 1-2 has a linear strain $= \partial u/\partial x$.

Because, Strain on 1-2 = [(1-6) – (1-2)]/1-2

Similarly the side 1-5 has a strain $= \partial v/\partial y$

Now consider the angular strain (shear strain) on 1-6 and 1-5

Shear strain on 1-6 = $\partial v/\partial x$

Similarly, shear strain on 1-5 = $\partial u/\partial y$

Total shear strain = $\gamma_{xy} = (\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})$

Now consider the rotation of 1-6 and 1-5:

We can write the total rotation = $\omega_{xy} = 1/2(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x})$

Here we consider anticlockwise rotation as positive

We are interested in transforming the strains along the given axes onto new axes which are inclined with the original axes at an angle. This may be similar to a situation in which we rotate the object itself through an angle and want to obtain the strains on the rotated object. The state of strains on an object subjected to stress can be represented with normal and shear strains. Only small strains can be transformed because for large strains, large angle changes may be involved.

While considering strain transformations, we consider a particular case, namely plane strain.

Plane strain condition is one in which the normal and shear strains along one of the three axes are zero.

$\varepsilon_z = 0$, $\gamma_{xz} = 0$, $\gamma_{yz} = 0$

The stress transformation equations derived for plane stress condition can also be applied for a condition of stress in which σ_z is also present. This is because σ_z is absent in the equilibrium equations.

This means that we can use the same transformation equations derived for stress for plane strain condition as well.

Normal strain is given as $\varepsilon_{xx} = \frac{\partial u}{\partial x}$,$\varepsilon_{xy} = \frac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})$.

Shear displacement is split into strain and rotation. Shear involves both displacement and rotation.

Shear strain is given as:

$\gamma_{xy} = (\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})$This strain is called engineering shear strain.

1.2 Strain transformations:

Let us assume plane strain condition.

We can write the strain transformations similar to stress transformations using direction cosines.

$$\varepsilon_x' = l_{x'x}^2\varepsilon_x + l_{x'y}^2\varepsilon_y + l_{x'x}l_{x'y}\gamma_{xy}$$

Similarly,

$$\gamma_{x'y'} = 2\varepsilon_x l_{x'x}l_{y'x} + 2\varepsilon_y l_{x'y}l_{y'y} + \gamma_{xy}(l_{x'x}l_{y'x} + l_{y'x}l_{x'y})$$

Here, $l_{x'x} = cos\theta \; and \; l_{x'y} = sin\theta$

Strain transformation equations for plane strain condition can be written as:

$$\varepsilon_{x_1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\frac{\gamma_{x_1y_1}}{2} = \frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

The principal normal strains are given as:

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

Principal shear strain is:

$$\frac{\gamma_{max}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

Minimum shear strain is of same magnitude as above but negative.

The similarity between plane stress transformation equations and plane strain transformation equations can be noted.

AT a point in a body, the principal stress and principal strain have the same direction.

1.3 Mohr's circle for strain:

Mohr's circle for strain is similar to that for stress. It is given below:

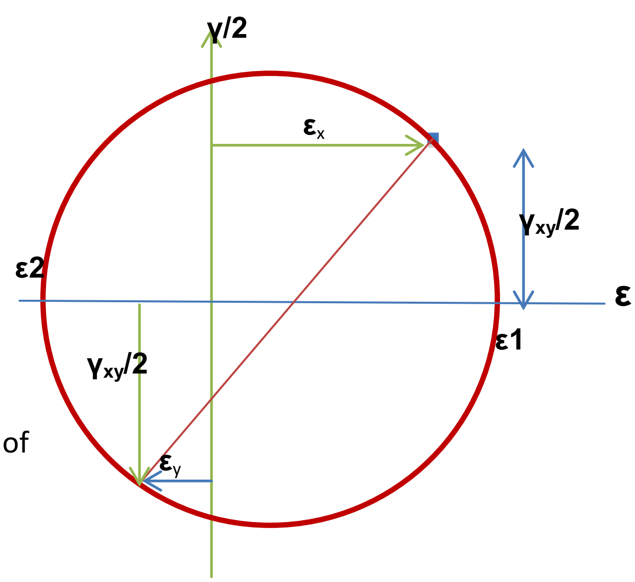

Fig. 2.3.1: Mohr's circle for plane strain

Note: The principal normal and principal shear stresses as well as strains are oriented at an angle of 45^{o} with respect to each other.

1.4 Equations of equilibrium of forces:

We can consider a plane of side dx and dy. Let us consider the equilibrium of forces on this plane:

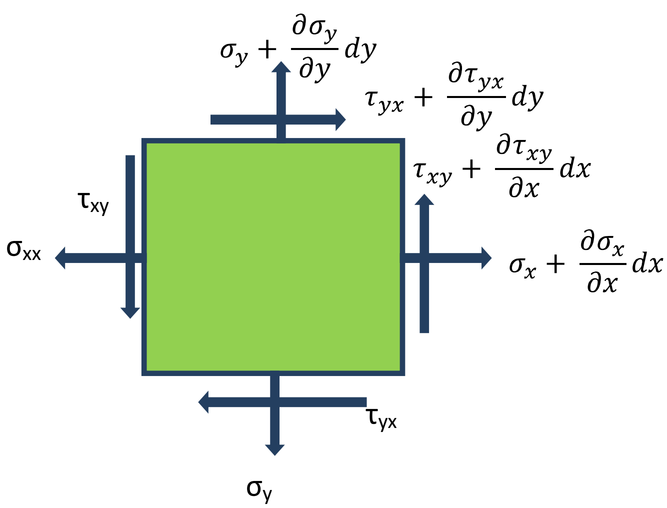

Fig. 2.3.1: Equilibrium of forces on a plane

The equilibrium equations are now written for biaxial stress as:

Along x direction, the force balance gives:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0$$

Along y direction:

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} = 0$$

We have two equations with 4 unknowns.

In order to solve these equations, we need two more equations.

These equations are called compatibility equations.

They are the equivalent stress and equivalent strains, also called effective stress and effective strain

Effective stress is given as:

$\bar{\sigma} = \frac{3}{\sqrt{2}}\tau_{oct}$ where τ_{oct} *is octahedral shear stress, given by*:

$$\tau_{oct} = \frac{1}{3}[\left(\sigma_x - \sigma_y\right)^2 + \left(\sigma_y - \sigma_z\right)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2)]$$

Or

$$\bar{\sigma} = \frac{1}{3}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

This equation is in terms of principal stresses.

The effective stress for uniaxial stress is simply equal to the yield strength.

We can define the effective strain as:

$$\bar{\varepsilon} = \sqrt{2}\gamma_{oct}$$

Or

$$\bar{\varepsilon} = \frac{\sqrt{2}}{3}[(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2]$$

In uniaxial stress, we can prove that the effective strain

$$\bar{\varepsilon} = longitudinal\ strain$$

Source:
http://nptel.ac.in/courses/112106153/7