Stability of Homogeneous Smart Beams Based on the First Order Shear Deformation Theory Located on a Continuous Elastic Foundation

A. R. Nezamabadi, and M. Karami Khorramabadi

***Abstract*—**This paper studies stability of homogeneous beams with piezoelectric layers subjected to axial load that is simply supported at both ends lies on a continuous elastic foundation. The displacement field of beam is assumed based on first order shear deformation beam theory. Applying the Hamilton's principle, the governing equation is established. The influences of applied voltage, dimensionless geometrical parameter and foundation coefficient on the stability of beam are presented. To investigate the accuracy of the present analysis, a compression study is carried out with a known data.

***Keywords*—**Stability, Homogeneous beam- Piezoelectric layer.

I. Introduction

THE applications of the smart materials have drawn attention in aerospace engineering, civil engineering, mechanical and even bio-engineering. The analysis of a coupled piezoelectric structure has recently been keenly researched because piezoelectric materials are more extensively used either as actuators or sensors. Examples include the analytical modelling and behaviour of a beam with surface-bonded or embedded piezoelectric sensors and actuators [1–3], and the use of piezoelectric materials in composite laminates and for vibration control [4]. The use of finite element method in the analysis of piezoelectric coupled structures has been studied [5–8]. Crawley and de Luis [9] developed the analytical model for the static and dynamic response of a beam structure with segmented piezoelectric actuators either bonded or embedded in a laminated composite. LaPeter and Cudney [10] proposed an analytic model for the segmented piezoelectric actuators bonded on a beam or a plate, and found the equivalent forcing functions of the actuators. The piezoelectric bimorph column structures were used as sensing elements. Dobrucki and Pruchnicki [11] presented an analysis theory of an axisymmetric piezoelectric bimorph. They also described a sensing theory for using the axisymmetric piezoelectric bimorph. Chandrashekhara and Bhatia [12] developed a finite element model for the active buckling control of laminated composite plates with surface bonded or embedded piezoelectric sensors that are either continuous or segmented. The dynamic buckling behavior of the laminated plate subjected to a linearly increasing compression load is investigated in their work. Chase and Bhashyam [13] derived optimal design equations to actively stabilize laminated plates loaded in excess of the critical buckling load using a large number of sensors and actuators.

To the author's knowledge, there is no analytical solution available in the open literatures for stability of homogeneous beams with piezoelectric layers subjected to axial load that is simply supported at both ends lies on a continuous elastic foundation. In the present work, stability of homogeneous beams with piezoelectric layers subjected to axial load that is simply supported at both ends lies on a continuous elastic foundation studied. Appling the Hamilton's principle, the equilibrium equations of beam are derived and solved. The effects of applied voltage, dimensionless geometrical parameter and foundation coefficient on the critical buckling load of beam are presented. To investigate the accuracy of the present analysis, a compression study is carried out with a known data.

II. Formulation

The formulation that is presented here is based on the assumptions of first order shear deformation beam theory. Based on this theory, the displacement field can be written as [14]:

$$\begin{aligned} u(x,z) &= z\phi(x) \\ w(x,z) &= w_0(x,z) \end{aligned} \tag{1}$$

In view of the displacement field given in Eqs. (1), the strain displacement relations are given by [14]:

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u}{\partial x} = z\frac{d\phi}{dx} \\ \gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \phi + \frac{dw}{dx} \end{aligned} \tag{2}$$

Consider a homogeneous beam with piezoelectric actuators and rectangular cross-section as shown in Fig. 1. The thickness, length, and width of the beam are denoted, respectively, by $h, L,$ and b. Also, h_T and h_B are the thickness of top and bottom of piezoelectric actuators, respectively. The $x-y$ plane coincides with the midplane of the beam and the $z-$axis located along the thickness direction.

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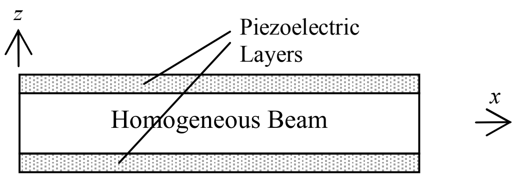

Fig. 1 Schematic of the problem studied

The Young's modulus E and the Poisson's ratio ν are assumed to be constant. The constitutive relations for homogeneous beam with piezoelectric layers are given by [15]:

$$\sigma_{xx} = Q_{11}\varepsilon_{xx} - e_{31}E_z$$
$$\sigma_{xz} = Q_{55}\gamma_{xz} - e_{15}E_x \tag{3}$$

Where

$$E_i = \frac{V}{h_i} \tag{4}$$

where $\sigma_{xx}, \sigma_{xz}, Q_{11}$ and Q_{55} are the normal ,shear stresses and plane stress-reduced stiffnesses and e_{31}, e_{15} are piezoelectric elastic stiffnesses respectively. Also, u and w are the displacement components in the $x-$ and $z-$directions, respectively.
The potential energy can be expressed as [14]:

$$U = \frac{1}{2}\int_v (\sigma_{xx}\varepsilon_{xx} + \sigma_{xz}\gamma_{xz})\,dv \tag{5}$$

Substituting Eqs. (2)-(3) into Eq. (5) and neglecting the higher-order terms, we obtain

$$U = \frac{1}{2}\int_v [(Q_{11}\left(z\frac{d\phi}{dx}\right) - e_{31}E_z)(z\frac{d\phi}{dx}) + (Q_{55}(\phi + \frac{dw}{dx}) - e_{15}E_x)(\phi + \frac{dw}{dx})]dv \tag{6}$$

The width of beam is assumed to be constant, which is obtained by integrating along y over v. Then Eq. (6) becomes

$$U = \frac{b}{2}\int_0^L [D\left(\frac{d\phi}{dx}\right)^2 + \frac{A}{2(1+\nu)}(\phi^2 + (\frac{dw}{dx})^2 + 2\phi\frac{dw}{dx})]dx - \frac{b}{2}\int_0^L \int_{-h_B-\frac{h}{2}}^{h_T+\frac{h}{2}} (ze_{31}E_z\frac{d\phi}{dx} + e_{15}E_x\varphi + e_{15}E_x\frac{dw}{dx})dzdx \tag{7}$$

where

$$A = \int_{-h_B-\frac{h}{2}}^{h_T+\frac{h}{2}} Q_{55}dz$$
$$D = \int_{-h_B-\frac{h}{2}}^{h_T+\frac{h}{2}} z^2 Q_{11}dz \tag{8}$$

Where A and D are the shear rigidity and flexural rigidity respectively. Note that, no residual stresses due to the piezoelectric actuator are considered in the present study and the extensional displacement is neglected. Thus, the potential energy can be written as:

$$U = \frac{b}{2}\int_0^L [D\,(\frac{d\phi}{dx})^2 + A(\phi^2 + (\frac{dw}{dx})^2 + 2\phi\frac{dw}{dx} - e_{31}(h_T V_T + h_B V_B)\frac{d\phi}{dx} - e_{15}(V_T + V_B)(\phi + \frac{dw}{dx})]dx \tag{9}$$

where V_T and V_B are the applied voltages on the top and bottom actuators respectively. The beam is subjected to the axial compressive loads, P as shown in Fig. 2.

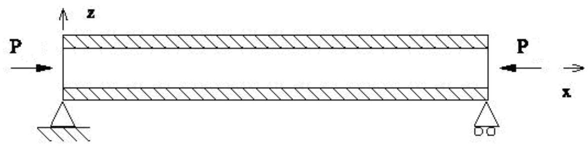

Fig. 2 Simply supported beam under periodic loads

The work done by the axial compressive load can be expressed as [14]:

$$W = \frac{1}{2}\int_0^L P\left(\frac{\partial w}{\partial x}\right)^2 dx \tag{10}$$

We apply the Hamilton's principle to derive the equilibrium equations of beam, that is [14]:

$$\delta\int_0^t (T - U + W)\,dt = 0 \tag{11}$$

Substitution from Eqs. (9) and (10) into Eq. (11) leads to the following equilibrium equations of the the homogeneous beam with piezoelectric layers based on first order shear deformation theory. Assume that a homogeneous beam with piezoelectric actuators that is simply supported at both ends lies on a continuous elastic foundation, whose reaction at every point is proportional to the deflection (Winkler foundation). The equilibrium equation of the homogeneous beam with piezoelectric layers based on first order shear deformation theory located on a continuous elastic foundation subjected to a axial compressive load is obtained from equilibrium equations by the addition of ηw for the foundation reaction as

$$(P-bA)\frac{d^2w}{dx^2}+bA(\frac{d\phi}{dx})=0$$
$$A(\phi+\frac{dw}{dx})+2e_{15}V_T+2D(\frac{d^2\phi}{dx^2})=0 \qquad (12)$$

where η is the foundation coefficient.

III. Stability Analysis

The boundary conditions for the pin-ended homogeneous beam are given by:

$$w=\frac{d^2w}{dx^2}=\frac{d\phi}{dx}, \quad \text{at} \quad x=0 \quad \text{and} \quad x=L \qquad (13)$$

Substituting Eq. (13) into (12) and neglecting the foundation coefficient and piezoelectric effect, the critical buckling load of a homogeneous beam based on first order shear deformation theory will be derived, that is:

$$p_{cr}=\frac{(\frac{\pi}{L})^2\frac{bh^3Q_{11}}{12}}{1+(\frac{L}{\pi})^2\frac{12Q_{55}}{bh^2Q_{11}}} \qquad (14)$$

The above equation has been reported by Wang and Reddy [14].

IV. Numerical Results

The stability of homogeneous beams with piezoelectric layers subjected to axial load that is simply supported at both ends lies on a continuous elastic foundation are studied in this paper. It is assumed that both the top and bottom piezoelectric layers have the same thickness; $h_T = h_B$ and the same voltages are applied to both actuators. The material properties of the beam are listed in Table I.

TABLE I
Material Properties

Property	Piezoelectric layer	Homogeneous layer
Young's modulus E (GPa)	63	223.95
Poisson's ratio ν	0.3	0.3
Length L (m)	0.3	0.3
Thickness h (m)	0.00005	0.01
Density ρ (Kgm^{-3})	7600	8900
Piezoelectric constant e_{31}, e_{15} (Cm^{-2})	17.6	-

The Poisson's ratio is chosen to be 0.3 for both materials. The variation of critical buckling loads for homogeneous beam versus η is shown in Table II and the variation of critical buckling loads for homogeneous beam versus h/L for different applied voltage is shown in Fig. 3.

TABLE II
Variation Of The Critical Buckling Load of Homogeneous Beam with Piezoelectric Actuators Versus η

Foundation Coefficient (η)	Critical Buckling Load (P_{cr})
1000	33000N
2000	38450N
3000	42761N
4000	46398N

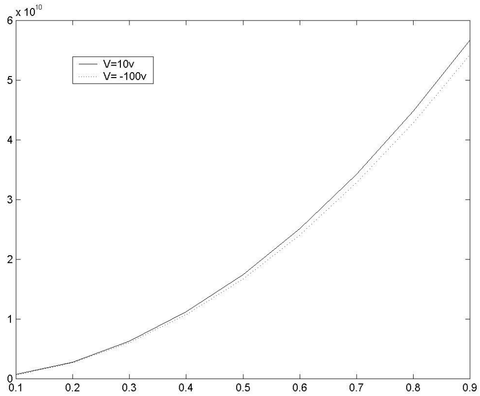

Fig. 3 Effect of Applied Voltage on the Critical Buckling Load of Homogeneous Beam with Piezoelectric Actuators

V. Conclusion

The stability of homogeneous beams with piezoelectric layers subjected to axial load that is simply supported at both ends lies on a continuous elastic foundation is studied. It is conclude that:

1- The piezoelectric actuators induce tensile piezoelectric force produced by applying negative voltages that significantly affect the stability of the homogeneous beams with piezoelectric actuators.
2- The critical buckling loads of homogeneous beams under axial compressive load generally increase with the increase of foundation coefficient η.
3- The accuracy of the first order shear deformation beam theory is more than the classical beam theory.

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