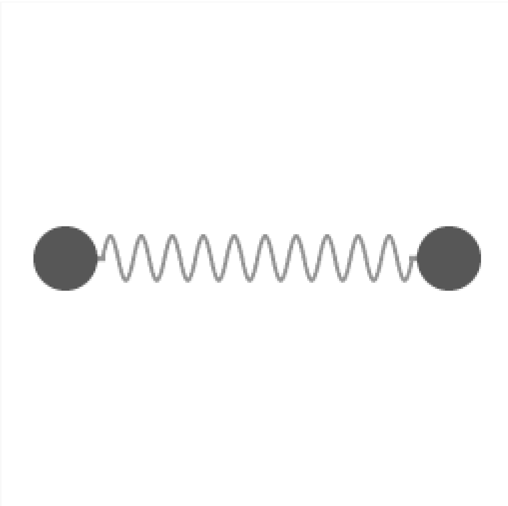
ROTATIONAL-VIBRATIONAL COUPLING

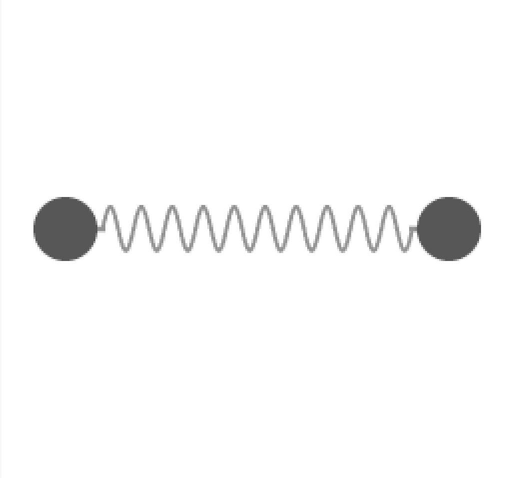


Picture 1. Animation
Two weights connected by a spring. There is a rotation around the common center of mass, and a coupled oscillation in radial distance and angular velocity.

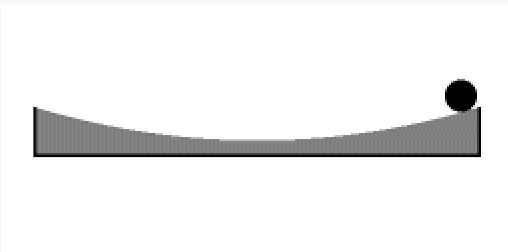
Rotational-vibrational coupling occurs when there is a 1:2 ratio of rotation frequency of an object and a natural internal vibration frequency. The animation on the right shows the simplest example of this phenomenon. The motion depicted in the animation is for the idealized situation that the force exerted by the spring is proportional to the amount of extension. Note that in this demonstration the spring isn't alternating between pulling and pushing, the spring is exerting a contracting force all the time; given the chance the idealized spring would contract all the way down to zero length. Also, since the animation keeps on looping, the animation depicts what would occur if there would not be any friction.

In molecular physics it is recognized that there is a coupling of rotational and vibrational energy-levels. In molecular physics rotational-vibrational coupling is also called **rovibronic coupling**and **Coriolis coupling**. The physics of actual diatomic molecules is more complicated than the example in the animation, but because of its simplicity the animation is useful for illustrating the basic principles.

Energy conversions in rotational-vibrational coupling.



Picture 2. Animation
The motion of animation 1 mapped in a coordinate system that is rotating at a constant angular velocity.



Picture 3. Animation
Harmonic oscillation; the restoring force is proportional to the distance to the center.

During the phase in the cycle that the spring pulls the two weights closer to the center of rotation the angular velocity increases; the centripetal force is doing work, converting strain energy that was stored in the spring to kinetic energy of the weights.
At some point contraction ends and the weights swing wide again. As the distance of the weights to the central axis of rotation increases kinetic energy is converted to strain energy of the spring. The angular velocity decreases during this phase, and from a certain point on there is a surplus of centripetal force. Eventually the surplus of centripetal force starts a new phase of contraction.

Analogy with harmonic oscillation

Animation 2 provides a clearer view on the oscillation of the angular velocity. The motion as seen from a rotating point of view looks remarkably regular and symmetrical. It is in fact very regular; I will come to that further on in the article.

Harmonic oscillation is a cyclic process of energy conversion. When a harmonic oscillation is at its midpoint then all the energy of the system is kinetic energy. When the harmonic oscillation is at the points furthest away from the midpoint all the energy of the system is potential energy. The total energy of the system is conserved, but its form is oscillating back and forth between kinetic energy and potential energy.

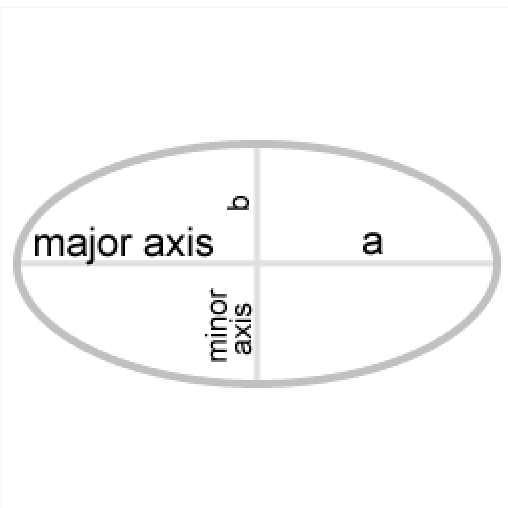
In the motion pattern depicted in animation 2 there is, just as in the case of simple harmonic oscillation, a back and forth conversion between kinetic energy and potential energy. When the spring is at its maximal extension then the potential energy is largest, when the angular velocity is at its maximum the kinetic energy is at its largest. The total energy of the system is conserved.

(With a real spring there is friction involved. With a real spring the vibration will be dampened and the final situation will be that the masses circle each other at a constant distance, with a constant tension of the spring. That is, in the final situation the ratio of kinetic energy to potential energy will be 1:1 .)

For a discussion of the role that momentum plays, see note on momentum

Mathematical derivation

Cartesian coordinates



Picture 4. Image

The motion of the circling masses is planar.
I'm applying the following simplifications: I'm taking the spring itself as being weightless, and I'm taking perfect spring; the centripetal force increases in a linear way as the spring is stretched out. That is, in this simplification the centripetal force is exactly proportional to the distance to the center of rotation. A centripetal force with this characteristic is called a **harmonic force**.

The motion of the weights in two dimensions of space can be decomposed in two harmonic oscillations, perpendicular to each other.

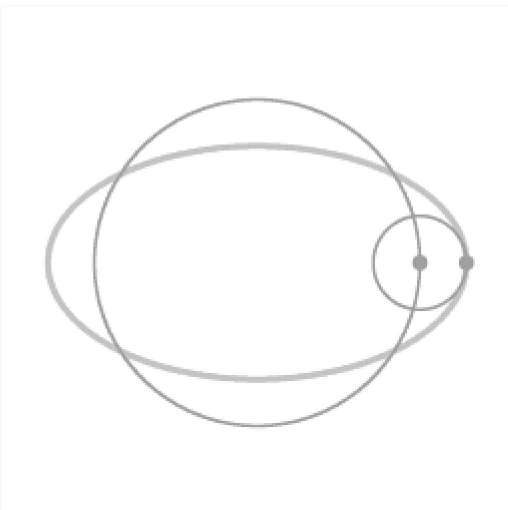
The following parametric equation of the position as a function of time describes the motion of the circling masses. The parametric equation provides a complete description: it describes the shape of the trajectory and the velocity at each point in time.

$$x = a\cos(\Omega t) \quad (1)$$

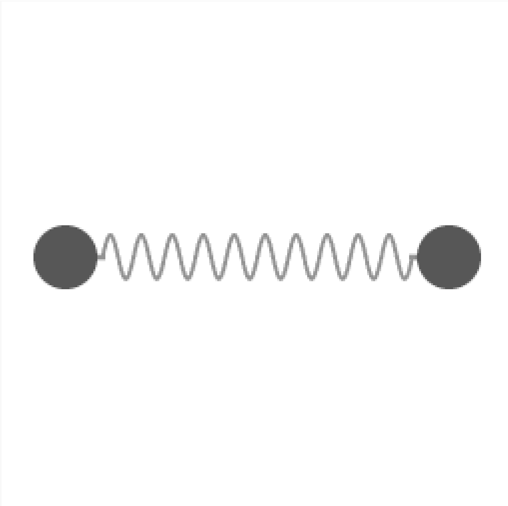
$$y = b\sin(\Omega t) \quad (2)$$

a half the length of the major axis
b half the length of the minor axis
Ω 360° divided by the duration of one revolution

In the articles on this site I'm using the greek capital Ω (Omega) to refer to a constant factor, a non-changing angular velocity of a system as a whole; when the small greek letter ω (omega) is used it refers to the instantaneous angular velocity of some object. The instantaneous angular velocity of some object may fluctuate, depending on the circumstances. In the case of a harmonic force the trajectory of the object can be expressed in terms of a function that contains a constant factor Ω.



Picture 5. Animation
An ellipse-shaped trajectory (due to a harmonic restoring force) can be seen as a circular motion (in this example counterclockwise), with the eccentricity as an epicircle (clockwise).



Picture 6. Animation
The motion of animation 1 mapped in a coordinate system that is rotating at a constant angular velocity.

The trajectory is excentric. To find exactly how the object moves as seen from a rotating point of view the parametric equation can be rearranged into the following two components:

$$x = \left(\frac{a+b}{2}\right)\cos(\Omega t) + \left(\frac{a-b}{2}\right)\cos(\Omega t)$$

$$y = \left(\frac{a+b}{2}\right)\sin(\Omega t) - \left(\frac{a-b}{2}\right)\sin(\Omega t)$$

Animation 5 depicts this rearrangement. There is an overall circular motion, combined with motion along an epi-circle. Both the motion along the overal circle (counterclockwise) and the motion long the epi-circle (clockwise) are *uniform* circular motion. This is a remarkable *symmetry* of motion under the influence of a harmonic force; the eccentricity itself can be thought of as a uniform circular motion.

Transformation to a coordinate system that is rotating with angular velocity Ω. does the following: it subtracts the overall circular motion, and what is left is the *eccentricity* of the elliptical trajectory. The center of the eccentricity is located at a distance of (a + b) / 2 from the main axis of rotation.

The transformation to a rotating coordinate system is of course to a particular one: the coordinate system in which the eccentricty of the ellipse-shaped trajectory is a circle around a fixed point. Here and everywhere else in this article (and everywhere else on this web site): whenever I refer to "transformation to a rotating coordinate system" I'm referring to the particular rotating coordinate system that matches the period of rotation.

$$x = \left(\frac{a-b}{2}\right)\cos(2\Omega t)$$

$$y = -\left(\frac{a-b}{2}\right)\sin(2\Omega t)$$

Animation 6 shows that as mapped in the rotating coordinate system the angular velocity along the epi-circle is 2Ω: it cycles*twice* for every cycle of the overall system. The radius of the eccentricity circle is (a - b) / 2.

I will call the radius of the eccentricity r_e.
We have as starting point the general formula for the centripetal acceleration a_c that is required to sustain uniform circular motion: $a_c = \omega^2 r$
We also have the following general relation between (instantaneous) linear velocity and angular velocity: $v = \omega r$.
As mapped in the co-rotating coordinate system the angular velocity along the epi-circle is 2Ω (Note that I use the capital letter Ω here). Hence the acceleration towards the center of the eccentricity is $(2\Omega)^2 r_e$.

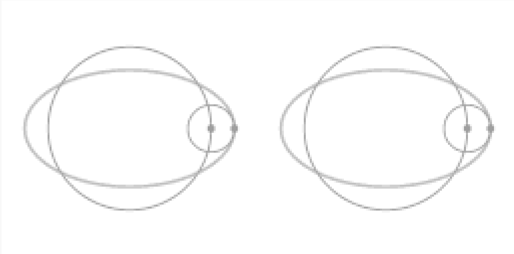
I will call the velocity with respect to the rotating system v_r. In this example the following relation is valid: $v_r = 2\Omega r_e$. This results in the following expression for the acceleration that corresponds to the motion along the epi-circle: $2\Omega v_r$ The direction of the acceleration is perpendicular to the direction of v_r.

Discussion using vector notation

As a given in this situation we have that the centripetal force is a harmonic force. (The strength of the centripetal force is proportional to the distance to the center of rotation.)

$$\vec{F} = -C\vec{r}$$

The set of all solutions to the above equation of motion consists of both circular trajectories and ellipse-shaped trajectories. All the solutions have the same period of revolution. This is a distinctive feature of motion under the influence of a harmonic force. There is no dependency on the *amplitude* of the trajectory; all trajectories take the same amount of time to complete a revolution.



Picture 7. Animation

When the motion is mapped in a rotating coordinate system the centrifugal term and the Coriolis term are added to the equation of motion.

$$
\begin{array}{ll}
centrifugal\ term & \Omega^2\vec{r} \\
coriolis\ term & 2(\vec{\Omega} \times \vec{v})
\end{array}
$$

Here, Ω is the angular velocity of the rotating coordinate system with respect to the inertial coordinate system. *v* is velocity of the moving object with respect to the rotating coordinate system. It is important to note that the centrifugal term only relates to the coordinate system, it does **not** relate to the motion of the object. The angular velocity of the object varies over time, but the Ω in the centrifugal term doesn't: it is the *uniform* angular velocity of the rotating *coordinate system*.

It's tempting to equate the centrifugal term with the amount of acceleration that an accelerometer registers when an object is pulling G's. It's true that in the case of motion along a perfectly circular trajectory the centrifugal term matches the G-count, but this comparison breaks down when the motion is not perfectly circular.

The Coriolis term is notated as a *vector cross product* which means (in this case) that the acceleration associated with the Coriolis term is at right angles to the direction of the velocity. As discussed above, the Coriolis term describes the acceleration that is associated with the *eccentricity* of the ellipse-shaped trajectory.

The centrifugal term and the Coriolis term are added to the equation of motion. In all, this gives the following three terms in the equation of motion for motion with respect to a coordinate system rotating with angular velocity Ω .

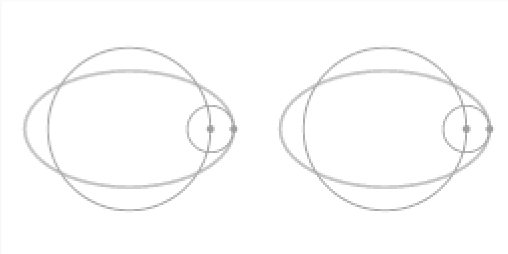
$$\vec{F} = -\,C\vec{r} + m\Omega^2\vec{r} + 2m(\vec{\Omega} \times \vec{v})$$

The angular velocity of the rotating coordinate system has been *matched* to the period of the oscillation; it follows that the vector of the centripetal force and the vector of the centrifugal term are at every distance to the center equal to each other in magnitude. Being opposite in direction the two terms drop away against each other.

It is only in very special circumstances that the vector representing the centripetal force and the vector that represents the centrifugal term can drop away against each other in the equation of motion. This is the case if and only if the centripetal force is a harmonic force.

There is the question of what it means to let the expression for the centripetal force and the centrifugal term drop away against each other. The centripetal force is a force, and the centrifugal term doesn't represent a force. That means that the two terms drop away against each other in a numerical sense only; it's merely a computational convenience.

Allowing the expressions for the centripetal force and the centrifugal term to drop away against each other leaves only the Coriolis term.


Picture 8. Animation

$$\vec{F} = 2m(\vec{\Omega} \times \vec{v})$$

The above equation of motion describes what the effect of the centripetal force is for the motion as mapped in a rotating coordinate system. Here, the Coriolis term describes a relation between velocity with respect to the rotating coordinate system and acceleration with respect to the rotating coordinate system. When the centripetal force pulls the weights closer to the central axis of rotation the angular velocity of the weights increases (and conversely, when the weights swing wide their angular velocity deacreases.) When the weights move slower than Ω they are pulled closer to the central axis of rotation (and conversely, when the move faster than Ω they tend to swing wide.)

As expected, the analysis using vector notation results in confirmation of the previous analysis: the spring is oscillating between doing positive work and doing negative work, and this accounts for the motion pattern.

Coriolis effect

Since the motion with respect to the rotating coordinate system is described with only the Coriolis term, it is quite fitting to refer to the motion with respect to the rotating coordinate system as the simplest and purest example of the *Coriolis effect*. This explains why in molecular physics and in engineering rotational-vibrational coupling is also referred to as **Coriolis coupling**.

The effect described in this article is the same for any direction with respect to the rotating coordinate system. This property of 'the same in all directions with respect to the rotating coordinate system' is especially significant in the phenomenon of inertial oscillation, that I discuss in the next article.

Note on momentum
In the section 'Energy conversions in rotational-vibrational coupling' the dynamics is followed by keeping track of the energy conversions, rather than looking at the angular momentum. The advantage of following the energy conversions is that it focusus on causality.

Important as momentum is, momentum deals with spatial symmetry of the laws of physics, rather than with a cause-to-effect relation. This can be shown with the example of a cannon being fired. When a cannon is fired, the projectile will shoot out of the barrel towards the target, and the barrel will recoil. It would be wrong to suggest that the projectile leaves the barrel at high velocity *because* of the recoil of the barrel. While recoil of the barrel will always occur, (codified as the principle of conservation of momentum) it is not a causal agent. The causal mechanism is in the energy conversions: the explosion of the gun powder converts potential chemical energy to the potential energy of a highly compressed gas. As the gas expands, its high pressure exerts a force on both the projectile and the interior of the barrel. It is through the action of that force that potential energy is converted to kinetic energy of both projectile and barrel.

Similarly, in the case of rotational-vibrational coupling, the increase of angular velocity on contraction is consistent with the principle of conservation of angular momentum, but that should not be confused with conservation of angular momentum being a causal agent. In the case of rotational-vibrational coupling, the causal agent is the force exerted by the spring. The spring is oscillating between doing work and doing negative work. (The work is taken to be negative when the direction of the force is opposite to the direction of the motion.)

Source : http://www.cleonis.nl/physics/phys256/coupling.php