Rheological Models

In this section, a number of one-dimensional linear viscoelastic models are discussed.

10.3.1 Mechanical (rheological) models

The word viscoelastic is derived from the words "viscous" + "elastic"; a viscoelastic material exhibits both viscous and elastic behaviour – a bit like a fluid and a bit like a solid. One can build up a model of linear viscoelasticity by considering combinations of the linear elastic spring and the linear viscous dash-pot[1]. These are known as **rheological models** or **mechanical models**.

The Linear Elastic Spring

The constitutive equation for a material which responds as a linear elastic spring of stiffness E is (see Fig. 10.3.1)

$$\varepsilon = \frac{1}{E}\sigma \tag{10.3.1}$$

The response of this material to a creep-recovery test is to undergo an instantaneous elastic strain upon loading, to maintain that strain so long as the load is applied, and then to undergo an instantaneous de-straining upon removal of the load.

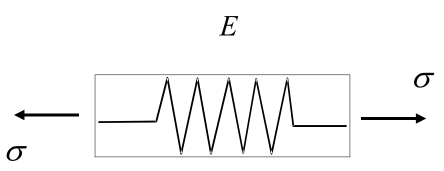

Figure 10.3.1: the linear elastic spring

The Linear Viscous Dash-pot

Imagine next a material which responds like a viscous dash-pot; the dash-pot is a piston-cylinder arrangement, filled with a viscous fluid, Fig. 10.3.2 – a strain is achieved by dragging the piston through the fluid. By definition, the dash-pot responds with a strain-rate proportional to stress:

$$\dot{\varepsilon} = \frac{1}{\eta}\sigma \tag{10.3.2}$$

where η is the **viscosity** of the material. This is the typical response of many **fluids**; the larger the stress, the faster the straining (as can be seen by pushing your hand through water at different speeds).

[1] a non-linear theory can be developed by including non-linear springs and dash pots

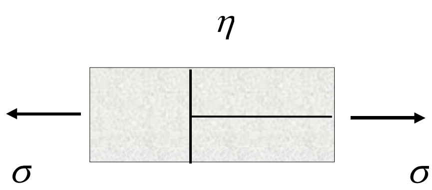

Figure 10.3.2: the linear dash-pot

The strain due to a suddenly applied load σ_o may be obtained by integrating the constitutive equation 10.3.2. Assuming zero initial strain, one has

$$\varepsilon = \frac{\sigma_o}{\eta} t \qquad (10.3.3)$$

The strain is seen to increase linearly and without bound so long as the stress is applied, Fig. 10.3.3. Note that there is no movement of the dash-pot at the onset of load; it takes time for the strain to build up. When the load is removed, there is no stress to move the piston back through the fluid, so that any strain built up is permanent. The slope of the creep-line is σ_o / η.

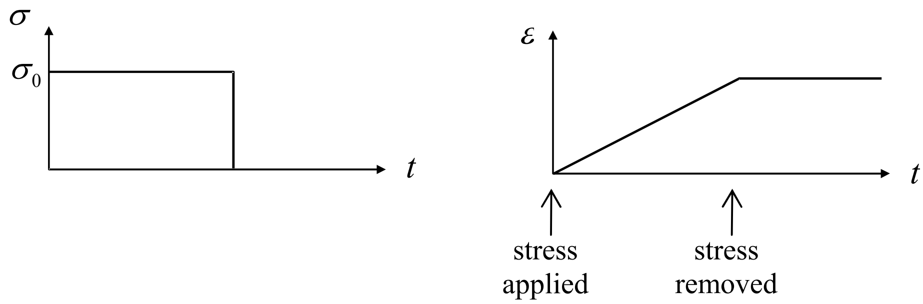

Figure 10.3.3: Creep-Recovery Response of the Dash-pot

The linear relationship between the stress and strain during the creep-test may be expressed in the form

$$\varepsilon(t) = \sigma_o J(t), \quad J(t) = \frac{t}{\eta} \qquad (10.3.4)$$

J here is called the **creep (compliance) function** ($J = 1/E$ for the elastic spring).

10.3.2 The Maxwell Model

Consider next a spring and dash-pot in series, Fig. 10.3.4. This is the **Maxwell model**. One can divide the total strain into one for the spring (ε_1) and one for the dash-pot (ε_2). Equilibrium requires that the stress be the same in both elements. One thus has the following three equations in four unknowns:

$$\varepsilon_1 = \frac{1}{E}\sigma, \qquad \dot{\varepsilon}_2 = \frac{1}{\eta}\sigma, \qquad \varepsilon = \varepsilon_1 + \varepsilon_2 \tag{10.3.5}$$

To eliminate ε_1 and ε_2, differentiate the first and third equations, and put the first and second into the third:

$$\boxed{\sigma + \frac{\eta}{E}\dot{\sigma} = \eta\dot{\varepsilon}} \qquad \textbf{Maxwell Model} \tag{10.3.6}$$

This constitutive equation has been put in what is known as **standard form** – stress on left, strain on right, increasing order of derivatives from left to right, and coefficient of σ is 1.

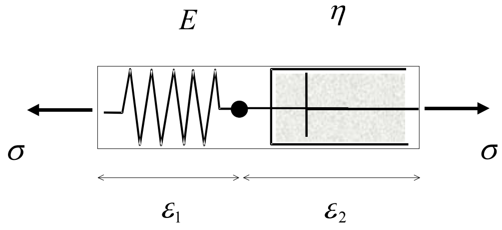

Figure 10.3.4: the Maxwell Model

Creep-Recovery Response

Consider now a creep test. Physically, when the Maxwell model is subjected to a stress σ_0, the spring will stretch immediately and the dash-pot will take time to react. Thus the initial strain is $\varepsilon(0) = \sigma_o / E$. Using this as the initial condition, an integration of 10.3.6 (with a zero stress-rate[2]) leads to

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[2] there is a jump in stress from zero to σ_o when the load is applied, implying an infinite stress-rate $\dot{\sigma}$. One is not really interested in this jump here because the corresponding jump in strain can be predicted from the physical response of the spring. One is more interested in what happens just "after" the load is applied. In that sense, when one speaks of initial strains and stress-rates, one means their values at 0^+, just after $t = 0$; the stress-rate is zero from 0^+ on. To be more precise, one can deal with the sudden jump in stress by integrating the constitutive equation across the point $t = 0$ as follows:

$$(E/\eta)\int_{-\Delta\tau}^{+\Delta\tau}\sigma(t)dt + \int_{-\Delta\tau}^{+\Delta\tau}\dot{\sigma}(t)dt = E\int_{-\Delta\tau}^{+\Delta\tau}\dot{\varepsilon}(t)dt$$

$$\rightarrow (E/\eta)\int_{-\Delta\tau}^{+\Delta\tau}\sigma(t)dt + \left[\sigma(+\Delta\tau) - \sigma(-\Delta\tau)\right] = E\left[\varepsilon(+\Delta\tau) - \varepsilon(-\Delta\tau)\right]$$

In the limit as $\Delta\tau \to 0$, the integral tends to zero (σ is finite), the values of stress and strain at 0^-, i.e. in the limit as $\Delta\tau \to 0$ from the left, are zero. All that remains are the values to the right, giving $\sigma(0^+) = E\varepsilon(0^+)$, as expected. One can deal with this sudden behaviour more easily using integral formulations or with the Laplace Transform (see §10.4, §10.5)

$$\dot{\varepsilon} = \frac{\sigma_o}{\eta} \quad \to \quad \varepsilon(t) = \frac{\sigma_o}{\eta} t + C$$
$$\to \quad \varepsilon(t) = \sigma_o \left(\frac{1}{\eta} t + \frac{1}{E} \right) \tag{10.3.7}$$

The creep-response can again be expressed in terms of a creep compliance function:

$$\varepsilon(t) = \sigma_o J(t) \quad \text{where} \quad J(t) = \frac{t}{\eta} + \frac{1}{E} \tag{10.3.8}$$

When the load is removed, the spring again reacts immediately, but the dash-pot has no tendency to recover. Hence there is an immediate elastic recovery σ_o / E, with the creep strain due to the dash-pot remaining. The full creep and recovery response is shown in Fig. 10.3.5.

The Maxwell model predicts creep, but not of the ever-decreasing strain-rate type. There is no anelastic recovery, but there is the elastic response and permanent strain.

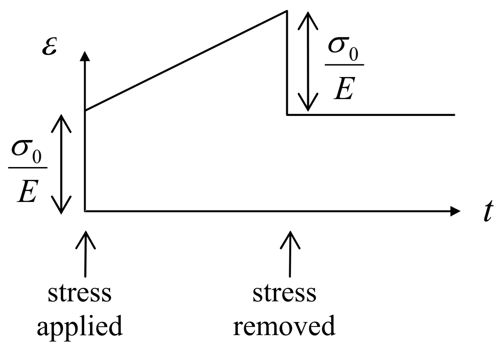

Figure 10.3.5: Creep-Recovery Response of the Maxwell Model

Stress Relaxation

In the stress relaxation test, the material is subjected to a constant strain ε_0 at $t = 0$. The Maxwell model then leads to {▲Problem 1}

$$\sigma(t) = \varepsilon_o E(t) \quad \text{where} \quad E(t) = E e^{-t/t_R}, \quad t_R = \frac{\eta}{E} \tag{10.3.9}$$

Analogous to the creep function J for the creep test, $E(t)$ is called the **relaxation modulus** function.

The parameter t_R is called the **relaxation time** of the material and is a measure of the time taken for the stress to relax; the shorter the relaxation time, the more rapid the stress relaxation.

10.3.3 The Kelvin (Voigt) Model

Consider next the other two-element model, the **Kelvin** (or **Voigt**) **model**, which consists of a spring and dash-pot in parallel, Fig. 10.3.6. It is assumed there is no bending in this type of parallel arrangement, so that the strain experienced by the spring is the same as that experienced by the dash-pot. This time,

$$\varepsilon = \frac{1}{E}\sigma_1, \quad \dot{\varepsilon} = \frac{1}{\eta}\sigma_2, \quad \sigma = \sigma_1 + \sigma_2 \tag{10.3.10}$$

where σ_1 is the stress in the spring and σ_2 is the dash-pot stress. Eliminating σ_1, σ_2 leaves the constitutive law

$$\boxed{\sigma = E\varepsilon + \eta\dot{\varepsilon}} \quad \textbf{Kelvin (Voigt) Model} \tag{10.3.11}$$

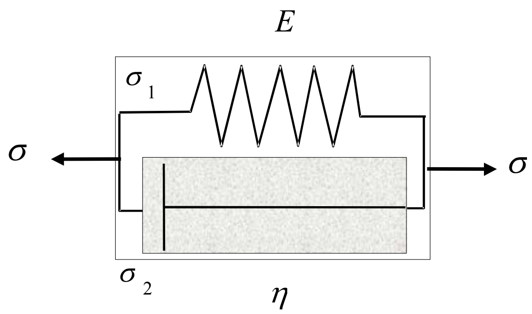

Figure 10.3.6: the Kelvin (Voigt) Model

Creep-Recovery Response

If a load σ_0 is applied suddenly to the Kelvin model, the spring will want to stretch, but is held back by the dash-pot, which cannot react immediately. Since the spring does not change length, the stress is initially taken up by the dash-pot. The creep curve thus starts with an initial slope σ_o / η.

Some strain then occurs and so some of the stress is transferred from the dash-pot to the spring. The slope of the creep curve is now σ_2 / η, where σ_2 is the stress in the dash-pot, with σ_2 ever-decreasing. In the limit when $\sigma_2 = 0$, the spring takes all the stress and thus the maximum strain is σ_o / E.

Solving the first order non-homogeneous differential equation 10.3.11 with the initial condition $\varepsilon(0) = 0$ gives

$$\varepsilon(t) = \frac{\sigma_o}{E}\left(1 - e^{-(E/\eta)t}\right) \tag{10.3.12}$$

which agrees with the above physical reasoning; the creep compliance function is now

$$J(t) = \frac{1}{E}\left(1 - e^{-t/t_R}\right), \quad t_R = \frac{\eta}{E} \tag{10.3.13}$$

The parameter t_R, in contrast to the relaxation time of the Maxwell model, is here called the **retardation time** of the material and is a measure of the time taken for the creep strain to accumulate; the shorter the retardation time, the more rapid the creep straining.

When the Kelvin model is unloaded, the spring will want to contract but again the dash pot will hold it back. The spring will however eventually pull the dash-pot back to its original zero position given time and full recovery occurs.

Suppose the material is unloaded at time $t = \tau$. The constitutive law, with zero stress, reduces to $0 = E\varepsilon + \eta\dot{\varepsilon}$. Solving leads to

$$\varepsilon(t) = Ce^{-(E/\eta)t} \tag{10.3.14}$$

where C is a constant of integration. The t here is measured from the point where "zero load" begins. If one wants to measure time from the onset of load, t must be replaced with $t - \tau$. The strain at $t = \tau$ is $\varepsilon(\tau) = (\sigma_o / E)\left(1 - e^{-(E/\eta)\tau}\right)$. Using this as the initial condition, one finds that

$$\varepsilon(t) = \frac{\sigma_o}{E} e^{-(E/\eta)t}\left(e^{(E/\eta)\tau} - 1\right), \quad t > \tau \tag{10.3.15}$$

The creep and recovery response is shown in Fig. 10.3.7. There is a transient-type creep and anelastic recovery, but no instantaneous or permanent strain.

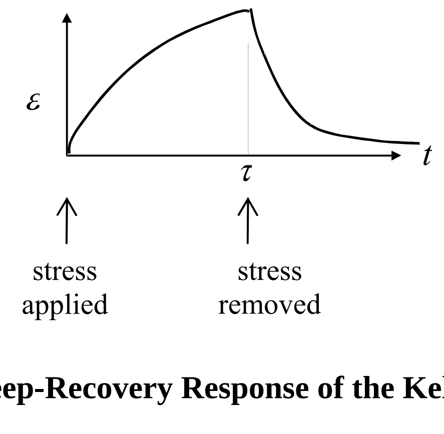

Figure 10.3.7: Creep-Recovery Response of the Kelvin (Voigt) Model

Stress Relaxation

Consider next a stress-relaxation test. Setting the strain to be a constant ε_0, the constitutive law 10.3.11 reduces to $\sigma = E\varepsilon_0$. Thus the stress is taken up by the spring and is constant, so there is in fact no stress relaxation over time. Actually, in order that the Kelvin model undergoes an instantaneous strain of ε_0, an infinite stress needs to be applied, since the dash-pot will not respond instantaneously to a finite stress[3].

10.3.4 Three – Element Models

The Maxwell and Kelvin models are the simplest, two-element, models. More realistic material responses can be modelled using more elements. The four possible three-element models are shown in Fig. 10.3.8 below. The models of Fig. 10.3.8a-b are referred to as "solids" since they react instantaneously as elastic materials and recover completely upon unloading. The models of Figs. 10.3.8c-d are referred to as "fluids" since they involve dashpots at the initial loading phase and do not recover upon unloading.

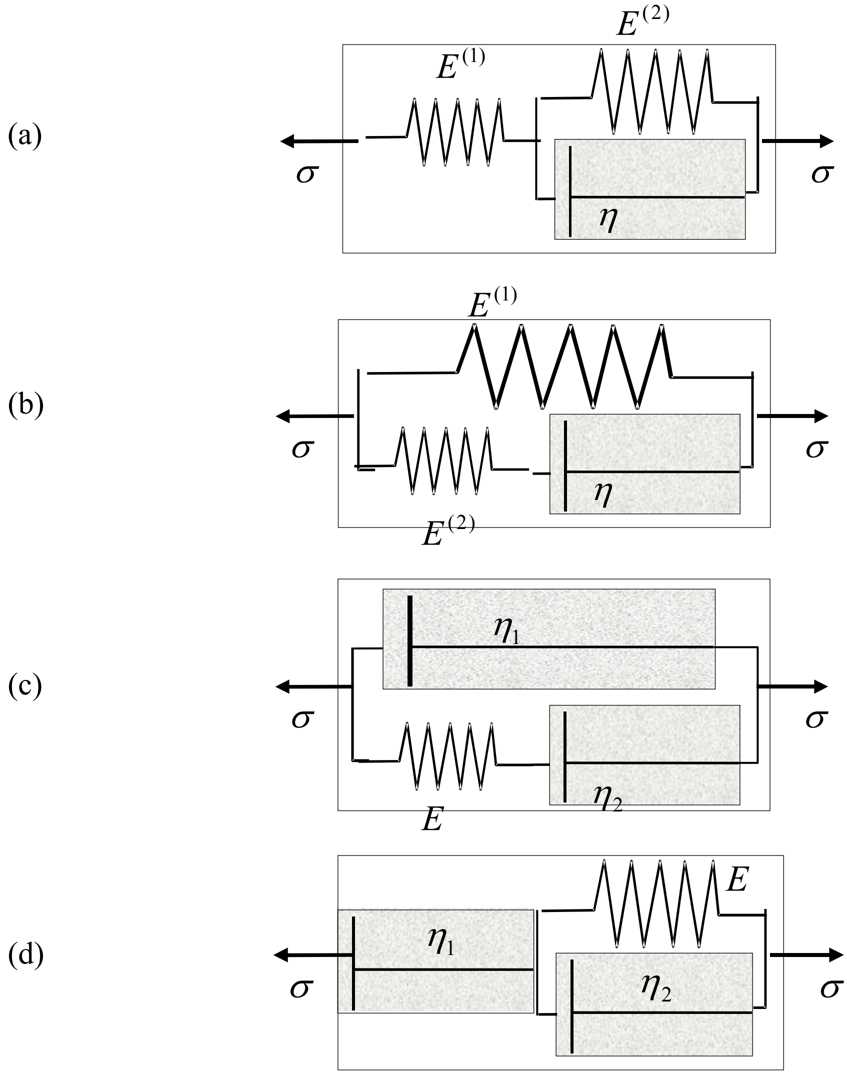

Figure 10.3.8: Three-element Models: (a) Standard Solid I, (b) Standard Solid II, (c) Standard Fluid I, (d) Standard Fluid II

[3] the stress required is $\sigma(0) = \eta\varepsilon_0\delta(t)$, where $\delta(t)$ is the Dirac delta function (this can be determined using the integral representations of §10.4

The differential constitutive relations for the Maxwell and Kelvin models were not difficult to derive. However, even with three elements, deriving them can be a difficult task. This is because one needs to eliminate variables from a set of equations, one or more of which is a differential equation (for example, see 10.3.5). The task is more easily accomplished using integral formulations and the Laplace transform, which are discussed in §10.4-§10.5.

Only results are given here: the constitutive relations for the four models shown in Fig. 10.3.8 are

$$\begin{array}{ll}
\text{(a)} & \sigma + \dfrac{\eta}{E_1 + E_2}\dot{\sigma} = \dfrac{E_1 E_2}{E_1 + E_2}\varepsilon + \dfrac{\eta E_1}{E_1 + E_2}\dot{\varepsilon} \\
\text{(b)} & \sigma + \dfrac{\eta}{E_2}\dot{\sigma} = E_1\varepsilon + \dfrac{\eta(E_1 + E_2)}{E_2}\dot{\varepsilon} \\
\text{(c)} & \sigma + \dfrac{\eta_2}{E}\dot{\sigma} = (\eta_1 + \eta_2)\dot{\varepsilon} + \dfrac{\eta_1\eta_2}{E}\ddot{\varepsilon} \\
\text{(d)} & \sigma + \dfrac{\eta_1 + \eta_2}{E}\dot{\sigma} = \eta_1\dot{\varepsilon} + \dfrac{\eta_1\eta_2}{E}\ddot{\varepsilon}
\end{array} \tag{10.3.16}$$

The response of these models can be determined by specifying stress (strain) and solving the differential equations 10.3.16 for strain (stress).

10.3.5 The Creep Compliance and the Relaxation Modulus

The creep compliance function and the relaxation modulus have been mentioned in the context of the two-element models discussed above. More generally, they are defined as follows: the creep compliance is the strain due to unit stress:

$$\varepsilon(t) = \sigma_o J(t), \qquad \varepsilon(t) = J(t) \text{ when } \sigma_o = 1 \qquad \textbf{Creep Compliance} \tag{10.3.17}$$

The relaxation modulus is the stress due to unit strain:

$$\sigma(t) = \varepsilon_o E(t), \qquad \sigma(t) = E(t) \text{ when } \varepsilon_o = 1 \qquad \textbf{Relaxation Modulus} \tag{10.3.18}$$

Whereas the creep function describes the response of a material to a creep test, the relaxation modulus describes the response to a stress-relaxation test.

10.3.6 Generalized Models

More complex models can be constructed by using more and more elements. A complex viscoelastic rheological model will usually be of the form of the **generalized Maxwell model** or the **generalized Kelvin chain**, shown in Fig. 10.3.9. The generalized Maxwell model consists of N different Maxwell units in parallel, each unit with different parameter values. The absence of the isolated spring would ensure fluid-type behaviour, whereas the absence of the isolated dash-pot would ensure an instantaneous response. The generalised Kelvin chain consists of a chain of Kelvin units and again the isolated spring may be omitted if a fluid-type response is required.

In general, the more elements one has, the more accurate a model will be in describing the response of real materials. That said, the more complex the model, the more material parameters there are which need to be evaluated by experiment – the determination of a large number of material parameters might be a difficult, if not an impossible, task.

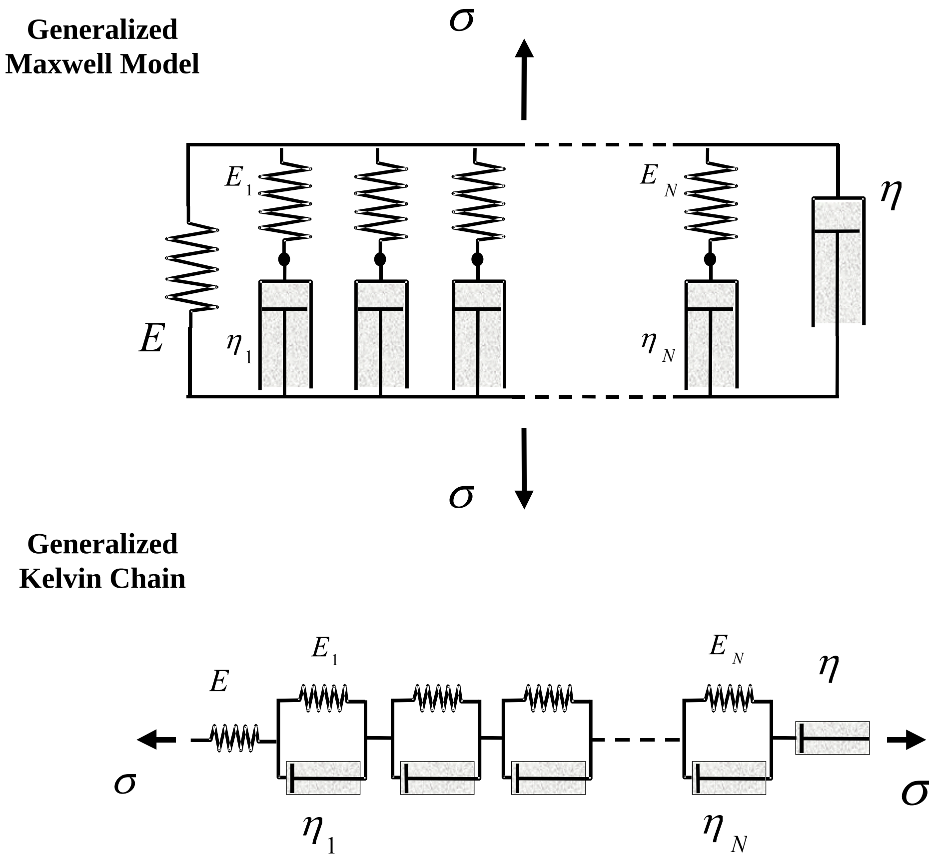

Figure 10.3.9: Generalised Viscoelastic Models

It is evident that, in general, a linear viscoelastic constitutive equation will be of the form

$$p_o\sigma + p_1\dot{\sigma} + p_2\ddot{\sigma} + p_3\dddot{\sigma} + p_4\sigma^{(IV)} + \cdots = q_o\varepsilon + q_1\dot{\varepsilon} + q_2\ddot{\varepsilon} + q_3\dddot{\varepsilon} + q_4\varepsilon^{(IV)} + \cdots \tag{10.3.19}$$

The more elements (springs/dashpots) one uses, the higher the order of the differential equation.

Eqn. 10.3.19 is sometimes written in the short-hand notation

$$\mathbf{P}\sigma = \mathbf{Q}\varepsilon \tag{10.3.20}$$

where $\mathbf{P}$ and $\mathbf{Q}$ are the linear differential operators

$$\mathbf{P} = \sum_{i=0}^{n} p_i \frac{\partial^i}{\partial t^i}, \qquad \mathbf{Q} = \sum_{i=0}^{n} q_i \frac{\partial^i}{\partial t^i} \tag{10.3.21}$$

A viscoelastic model can be created by simply entering values for the coefficients p_i, q_i, in 10.3.19, without referring to any particular rheological spring – dashpot arrangement. In that sense, springs and dashpots are not necessary for a model, all one needs is a differential equation of the form 10.3.19. However, the use of springs and dashpots is helpful as it gives one a physical feel for the way a material might respond, rather than simply using an abstract mathematical expression such as 10.3.19.

10.3.7 Retardation and Relaxation Spectra

Generalised models can contain many parameters and will exhibit a whole array of relaxation and retardation times. For example, consider two Kelvin units in series, as in the generalised Kelvin chain; the first unit has properties E_1, η_1 and the second unit has properties E_2, η_2. Using the methods discussed in §10.4-§10.5, it can be shown that he constitutive equation is

$$\sigma + \frac{\eta_1 + \eta_2}{E_1 + E_2}\dot{\sigma} = \frac{E_1 E_2}{E_1 + E_2}\varepsilon + \frac{E_1\eta_2 + E_2\eta_1}{E_1 + E_2}\dot{\varepsilon} + \frac{\eta_1\eta_2}{E_1 + E_2}\ddot{\varepsilon} \tag{10.3.22}$$

Consider the case of specified stress, so that this is a second order differential equation in $\varepsilon(t)$. The homogeneous solution is {▲Problem 3}

$$\varepsilon_h(t) = Ae^{-t/t_R^1} + Be^{-t/t_R^2} \tag{10.3.23}$$

where $t_R^1 = \eta_1 / E_1, t_R^2 = \eta_2 / E_2$ are the eigenvalues of 10.3.22. For a constant load σ_0, the full solution is {▲Problem 3}

$$\varepsilon(t) = \sigma_0\left[\frac{1}{E_1}\left(1 - e^{-t/t_R^1}\right) + \frac{1}{E_2}\left(1 - e^{-t/t_R^2}\right)\right] \tag{10.3.24}$$

Thus, whereas the single Kelvin unit has a single retardation time, Eqn. 10.3.13, this model has two retardation times, which are the eigenvalues of the differential constitutive equation. The term inside the square brackets is evidently the creep compliance of the model.

Note that, for constant strain, the model predicts a static response with no stress relaxation (as in the single Kelvin model).

In a similar way, for N units, it can be shown that the response of the generalised Kelvin chain to a constant load σ_0 is, neglecting the effect of the free spring/dashpot, of the form

$$\varepsilon(t)=\sigma_0\sum_{i=1}^{N}\frac{1}{E_i}\left(1-e^{-t/t_R^i}\right),\quad t_R^i=\frac{\eta_i}{E_i} \tag{10.3.25}$$

where E_i,η_i are the spring stiffness and dashpot viscosity of Kelvin element i, $i=1\ldots N$, Fig. 10.3.9. The response of real materials can be modelled by allowing for a number of different retardation times of different orders of magnitude, e.g. $t_R^i=\{\ldots,10^{-1},1,10^1,10^2,\ldots\}$.

If one considers many elements, Eqn. 10.3.25 can be expressed as

$$\varepsilon(t)=\sigma_0\sum_{i=1}^{N}\Delta\phi\left(t_R^i\right)\left(1-e^{-t/t_R^i}\right),\quad \Delta\phi\left(t_R^i\right)=\frac{1}{\eta_i}t_R^i \tag{10.3.26}$$

In the limit as $N\to\infty$, letting $d\phi=\left(d\phi/dt_R\right)dt_R$ one has, changing the dummy variable of integration from dt_R to λ, and letting $\varphi\left(t_R\right)=d\phi/dt_R$,

$$\varepsilon(t)=\sigma_0\int_0^\infty\phi(\lambda)\left(1-e^{-t/\lambda}\right)d\lambda \tag{10.3.27}$$

The representation 10.3.27 allows for a continuous retardation time, in contrast to the discrete times of the model 10.3.25. The function $\phi(\lambda)$ is called the **retardation spectrum** of the model. Different responses can be modelled by simply choosing different forms for the retardation spectrum.

An alternative form of Eqn. 10.3.27 is often used, using the fact that $d\lambda/d\left(\ln\lambda\right)=\lambda$:

$$\varepsilon(t)=\sigma_0\int_0^\infty\overline{\phi}(\lambda)\left(1-e^{-t/\lambda}\right)d\left(\ln\lambda\right) \tag{10.3.28}$$

where $\overline{\phi}=\lambda\phi$.

A similar analysis can be carried out for the Generalise Maxwell model. For two Maxwell elements in parallel, the constitutive equation can be shown to be

$$\sigma+\frac{E_1\eta_2+E_2\eta_1}{E_1E_2}\dot{\sigma}+\frac{\eta_1\eta_2}{E_1E_2}\ddot{\sigma}=\left(\eta_1+\eta_2\right)\dot{\varepsilon}+\frac{E_1+E_2}{E_1E_2}\eta_1\eta_2\ddot{\varepsilon} \tag{10.3.29}$$

Consider the case of specified strain, so that this is a second order differential equation in $\sigma(t)$. The homogeneous solution is, analogous to 10.3.23, {▲Problem 4}

$$\sigma_h(t)=Ae^{-t/t_R^1}+Be^{-t/t_R^2} \tag{10.3.30}$$

where again $t_R^1 = \eta_1 / E_1$, $t_R^2 = \eta_2 / E_2$, and are the eigenvalues of 10.3.29. For a constant strain ε_0, the full solution is {▲Problem 4}

$$\sigma(t) = \varepsilon_0 \left[E_1 e^{-t/t_R^1} + E_2 e^{-t/t_R^2} \right] \tag{10.3.31}$$

Thus, whereas the single Maxwell unit has a single relaxation time, Eqn. 10.3.9, this model has two relaxation times, which are the eigenvalues of the differential constitutive equation. The term inside the square brackets is evidently the relaxation modulus of the model.

By considering a model with an indefinite number of Maxwell units in parallel, each with vanishingly small elastic moduli ΔE_i, one has the expression analogous to 10.3.27,

$$\sigma(t) = \varepsilon_0 \int_0^\infty \vartheta(t_R) e^{-t/t_R} dt_R \tag{10.3.32}$$

and $\vartheta(t_R)$ is called the **relaxation spectrum** of the model.

To complete this section, note that, for the two Maxwell units in parallel, a constant stress σ_0 leads to the creep strain {▲Problem 5}

$$\begin{gathered} \varepsilon(t) = \sigma_0 \left[\frac{1}{E_1 + E_2} e^{-t/t_R} + \left(\frac{\eta_1 / E_1 + \eta_2 / E_2}{\eta_1 + \eta_2} - \frac{t_R}{\eta_1 + \eta_2} \right) \left(1 - e^{-t/t_R} \right) + \frac{t}{\eta_1 + \eta_2} \right], \\ t_R = \frac{\eta_1 \eta_2}{\eta_1 + \eta_2} \frac{E_1 + E_2}{E_1 E_2} \end{gathered} \tag{10.3.33}$$

Source: http://homepages.engineering.auckland.ac.nz/~pkel015/SolidMechanicsBooks/Part_I/BookSM_Part_I/10_Viscoelasticity/10_Viscoelasticity_03_Rheological.pdf