Equations of motion:

The matrix formulation even makes it possible to solve the system of differential equations using software that performs matrix computations. Equations (1-31) and (1-32) are therefore expressed as

$$m\ddot{x} + c\dot{x} + kx = F(t) \qquad (0\text{-}33)$$

where

$$[\mathbf{M}] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \qquad (0\text{-}34)$$

$$[\mathbf{D}] = \begin{bmatrix} C_1 + C_2 & -C_2 \\ -C_2 & C_2 + C_3 \end{bmatrix} \qquad (0\text{-}35)$$

$$[\mathbf{K}] = \begin{bmatrix} \copyright_1 + \copyright_2 & -\copyright_2 \\ -\copyright_2 & \copyright_2 + \copyright_3 \end{bmatrix} \qquad (0\text{-}36)$$

$$\vec{x}(t) = \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} \qquad (0\text{-}37)$$

$$\vec{F}(t) = \begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix} \qquad (0\text{-}38)$$

Once again, let the excitation forces and the particular solutions be expressed by rotating vectors

$$\mathbf{F}_1(t) = \hat{\mathbf{F}}_1 e^{i\mu t} \tag{0-39}$$

$$\mathbf{F}_2(t) = \hat{\mathbf{F}}_2 e^{i\mu t} \tag{0-40}$$

$$\mathbf{x}_{\mathbf{1p}}(t) = \hat{\mathbf{x}}_{\mathbf{1p}} e^{i\mu t} \tag{0-41}$$

$$\mathbf{x}_{2\mathbf{p}}(t) = \hat{\mathbf{x}}_{2\mathbf{p}} e^{i\mu t} \tag{0-42}$$

Putting (1-38,39, 40, 41) into (1-33) gives

$$-\mu^2 [\mathbf{M}] \cdot \{\hat{\mathbf{x}}_p\} + i\mu [\mathbf{D}] \cdot \{\hat{\mathbf{x}}_p\} + [\mathbf{K}] \cdot \{\hat{\mathbf{x}}_p\} = \{\hat{\mathbf{F}}\} \tag{0-43}$$

Solving to the homogeneous equations with the force vector $\vec{F}$ set equal to zero leads to the system's *eigenfrequencies*. Setting, moreover, the damping matrix equal to zero, in order to obtain the undamped eigenfrequencies, the latter are found to be real. Damping, on the other hand, brings about complex-valued eigenfrequencies; the complex values contain information on both the undamped eigenfrequencies and the system damping. The eigenfrequencies μ_1 and μ_2 are given by the homogeneous equation

$$\left.\begin{aligned} &([K]\ \omega^2\ [M]) = 0 \text{ which gives} \\ &\begin{vmatrix} k_1 + k_2 - \omega^2 & -k_2 \\ -k_2 & k_2 - \omega^2 m_2 \end{vmatrix} = 0 \\ &(k_1 + k_2 - \omega^2)\,(k_2 - \omega^2\, m_2) - k_2^2 = 0 \end{aligned}\right\} \tag{1.44}$$

The condition for the existence of solutions to (1-43) is that the *system determinant* is identically zero, i.e.,

Insert w_{n1} into $([K]-w^2[M])\{x\}=\{0\}$

By definition, $det([K]- w_{n1}{}^2 [M])=0$ (0-45)

For a two degree-of-freedom system, (1-43) has two solutions corresponding to two eigenfrequencies. A system with n degrees-of-freedom has n eigenfrequencies. The eigenfrequencies of the two degree-of-freedom system are

$$\mu_{1,2} = \sqrt{\frac{\copyright_1 + \copyright_2}{2m_1} + \frac{\copyright_2 + \copyright_3}{2m_2} \pm \sqrt{\frac{(\copyright_1 + \copyright_2)^2}{4m_1^2} + \frac{(\copyright_2 + \copyright_3)^2}{4m_2^2} + \frac{\copyright_2^2 - \copyright_1\copyright_2 - \copyright_1\copyright_3 - \copyright_2\copyright_3}{2m_1 m_2}}}$$

From linear algebra, it is known that there is an *eigenvector* corresponding to each *eigenvalue* (*eigenfrequency*). These eigenvectors are mutually independent (orthogonal), and contain information on how the system oscillates in the vicinity of their respective eigenfrequencies. The mode shapes, $\mathbf{x}_1$ and $\mathbf{x}_2$, are obtained by substituting the eigenfrequencies, i.e., the solutions of (1-38), into (1-37), yielding

$$-\mu_1^2[\mathbf{M}]\cdot\{\hat{\mathbf{x}}_1\} + [\mathbf{K}]\cdot\{\hat{\mathbf{x}}_1\} = \{0\} \qquad (0\text{-}46)$$

$$-\mu_2^2[\mathbf{M}]\cdot\{\hat{\mathbf{x}}_2\} + [\mathbf{K}]\cdot\{\hat{\mathbf{x}}_2\} = \{0\} \qquad (0\text{-}47)$$

Example 1.5 [1]

A method that provides a respectable amount of isolation of a vibrating machine is to use a so-called double-elastic mounting; see Figure 1-18.

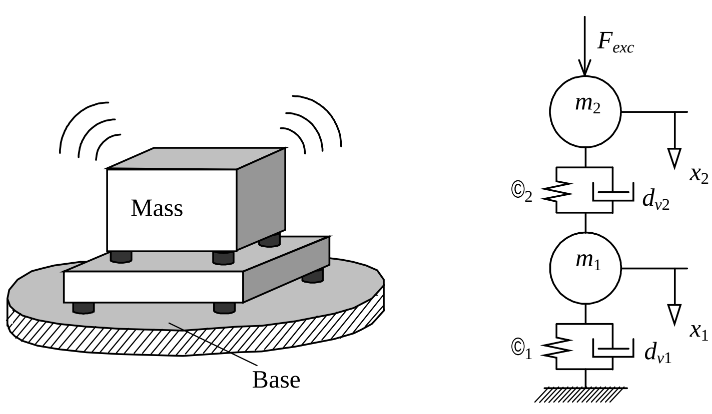

Figure 0.18 A double-layered elastic mounting can provide considerable isolation of a vibrating machine. [1]

Suppose that the vibrating machine is represented by a point mass m_2, and that its vibrations are generated by a harmonic excitation force F_{exc} with a circular frequency f . To reduce the resulting vibrations in the foundation, the machine is to be isolated by incorporating a spring – mass – spring system between the machine and the foundation, as illustrated in Figure 1-4. The parameters of the model are as follows:

m_1 = 100 kg, m_2 = 500 kg, $k_1 = 5{\cdot}10^6$ N/m, $k_2 = 1{\cdot}10^6$ N/m,

d_{v1} = 100 kg/s and d_{v2} = 200 kg/s.

a) Determine, for the isolated system, the
   (*i*) undamped eigenfrequencies. Which frequencies, i.e., vibration frequencies generated by the machine, is the mounted machine sensitive to?
   (*ii*) mode shapes.

Solution

a) In the first task, the undamped ([D] = 0) vibration isolation system's eigenfrequencies, and corresponding mode shapes, are determined.

(*i*) The undamped system's eigenfrequencies can be calculated with the aid of formula (1-39), with the third spring constant k_3 set equal to 0. Thus,

$$\mu_{1,2} = \cdots \approx \begin{cases} \sqrt{60343} \\ \sqrt{1657} \end{cases} \approx \begin{cases} 245.6 \\ 40.7 \end{cases} \text{ rad/s}$$

i.e.,

$$f_{1,2} \approx \begin{cases} 39.1 \\ 6.48 \end{cases} \text{ Hz}$$

(*ii*) The system's undamped mode shapes are the solutions to the homogeneous system, with the circular frequency f_c set to each of the eigenfrequencies in turn. The undamped ([D] = 0) homogeneous system of equations of motion is, with the assumed harmonic solution forms, exactly as given in (1-37). That, with values entered, becomes

$$\begin{cases} 6\cdot10^{6}\,\hat{\mathbf{x}}_1 - 1\cdot10^{6}\,\hat{\mathbf{x}}_2 - 100\mu_n^2\,\hat{\mathbf{x}}_1 = 0 \\ -1\cdot10^{6}\,\hat{\mathbf{x}}_1 + 1\cdot10^{6}\,\hat{\mathbf{x}}_2 - 500\mu_n^2\,\hat{\mathbf{x}}_2 = 0 \end{cases}, \quad n = 1, 2,$$

in which x_1 and x_2 (see Figure 1-16) indicate the coordinates of both masses. By multiplying the second of these equations by the factor

$$-\frac{1\cdot10^{6} - 100\mu_n^2}{6\cdot10^{6}},$$

and substituting in one of the two numerical values for the eigenfrequency, it becomes identical to the first equation. That implies that both equations are linearly dependent, in complete agreement with the theory. A linearly dependent system, with two unknowns, has an infinite number of solutions along a straight line in the x_1- x_2- plane. In order to solve the system, the equation of that line must be determined. Set the amplitude of x_1 to x_2 in the second equation of the set, and solve for the amplitude of x_2; that is found to be

$$\hat{\mathbf{x}}_2 = \frac{1\cdot10^{6}}{1\cdot10^{6} - 500\mu_n^2}\cdot\sim \; .$$

If the amplitude of x_1 has the value ω_n, then that of x_2 must have the value given by the formula above. Then, the eigenvector corresponding to eigenfrequency ω_n has the form

$$\vec{\therefore}_n = \sim\cdot\begin{Bmatrix} 1 \\ \dfrac{1\cdot10^{6}}{1\cdot10^{6} - 500\mu_n^2} \end{Bmatrix},$$

where α is an arbitrary constant. Thus, the eigenvector is a vector with a specified direction, but arbitrary length. The physical interpretation of the eigenvector's direction is the ratio between the amplitudes of motion of the two masses in a resonant oscillation.

Putting in the eigenfrequencies as described, with n set to 1 for the sake of convenience, provides an eigenvector or mode form for each,

$$\omega_n = \omega_1 : \qquad \vec{\cdot}_1 = \begin{Bmatrix} 1 \\ -0.034 \end{Bmatrix},$$

$$\omega_n = \omega_2 : \qquad \vec{\cdot}_2 = \begin{Bmatrix} 1 \\ 5.8 \end{Bmatrix}.$$

The interpretation of the first eigenvector, for example, is that if the system is excited by an excitation frequency near the first eigenfrequency, the system vibrates resonantly with the amplitude of the second mass 0.034 times that of the first. The minus sign, moreover, indicates that the masses move in opposite phase, i.e., in mutually opposing directions.

1.1.1 System with an arbitrary number of degrees-of-freedom

Real life systems are complex, they can bend, twist and elongate in axial direction, the mass is distributed, not discrete as assumed in the simple models, similarly, elasticity is distributed, there are no perfect springs without mass. In reality we have infinite degrees of freedom in a system, for convenience, we can model them as finite degrees of freedom systems. The methods of modeling have been refined over the years depending on the computational facilities available at respective times.

The results from the two degree-of-freedom system can be generalized to a system with an arbitrary number of masses cascaded, i.e., coupled in series, as in Figure 1-17.

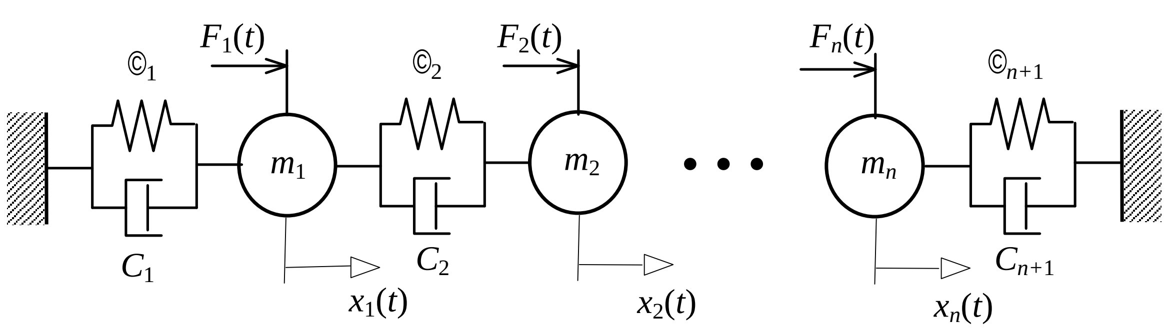

Figure 0-19 System with n cascaded masses

The equations of motion become;

$$m_1 \ddot{x}_1 + C_1 \dot{x}_1 + C_2\left(\dot{x}_1 - \dot{x}_2\right) + +©_1 x_1(t) + ©_2\left(x_1(t) - x_2(t)\right) = F_1(t) \tag{0-42}$$

$$m_2 \ddot{x}_2 + C_2\left(\dot{x}_1 - \dot{x}_2\right) + C_3 \dot{x}_2 - ©_2 (x_1(t) - x_2(t)) + ©_3\, x_2(t) = F_2(t) \tag{1.43}$$

$$m_{n-1} \ddot{x}_{n-1} + C_{1n-1}\left(\dot{x}_{n-2} - \dot{x}_{n-1}\right) + C_n\left(\dot{x}_{n-1} - \dot{x}_n\right) - ©_{n-1}(x_{n-2}(t) - x_{n-1}(t)) \tag{1-44}$$

$$+©_n\,(x_{n-1}(t) - x_n(t)) = F_{n-1}(t)$$

$$m_n \ddot{x}_n + C_{1n+1}\left(\dot{x}_n\right) + C_n\left(\dot{x}_n - \dot{x}_{n-1}\right) - ©_{n+1}\, x_n(t)$$

$$+©_n\,(x_n(t) - x_{n-1}(t)) = F_n(t) \tag{1.45}$$

The mass matrix, damping matrix, and stiffness matrix, respectively, become

$$[\mathbf{M}] = \begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & m_n \end{bmatrix} \tag{0-1}$$

$$[\mathbf{D}] = \begin{bmatrix} C_1 + C_2 & -C_2 & 0 & \cdot & \cdot & \cdot \\ -C_2 & C_2 + C_3 & -C_3 & 0 & \cdot & \cdot \\ 0 & -C_3 & \bullet & \bullet & \cdot & \cdot \\ \cdot & \cdot & \bullet & \bullet & \bullet & \cdot \\ \cdot & \cdot & 0 & -C_{n-1} & C_{n-1} + C_n & -C_n \\ \cdot & \cdot & \cdot & 0 & -C_n & C_n + C_{n+1} \end{bmatrix} \tag{0-2}$$

$$[\mathbf{K}]=\begin{bmatrix} ©_1+©_2 & -©_2 & 0 & \cdot & \cdot & \\ -©_2 & ©_2+©_3 & -©_3 & 0 & \cdot & \\ 0 & -©_3 & \bullet & \bullet & \cdot & \\ \cdot & \cdot & \bullet & \bullet & \bullet & \\ \cdot & \cdot & 0 & -©_{n-1} & ©_{n-1}+©_n & - \\ \cdot & \cdot & \cdot & 0 & -©_n & ©_n \cdot \end{bmatrix} \qquad (0\text{-}3)$$

where non-zero elements not shown in the equations are marked with a •, and zero-valued elements are marked with a ·. One can even allow masses to be coupled in parallel, as in Figure 1-18.

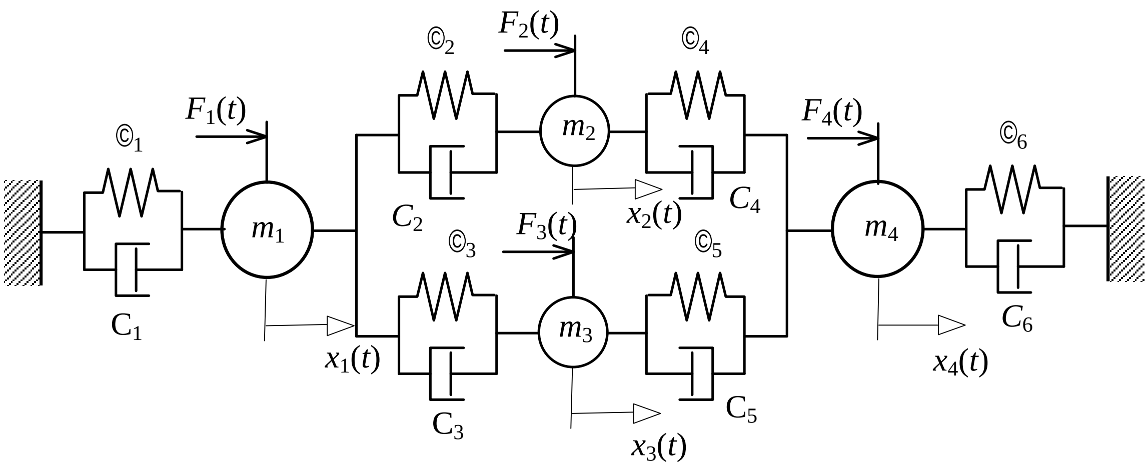

Figure 0-19 System with parallel coupling.

The equations of motion (1.49) to (1.52) are now;

$$m_1\ddot{x}_1+C_1\dot{x}_1+C_2\left(\dot{x}_1-\dot{x}_2\right)++©_1x_1(t)+©_2\left(x_1(t)-x_2(t)\right)+©_3\left(x_1(t)-x_3(t)\right)=F_1(t)$$

$$m_2\ddot{x}_2 + C_2\left(\dot{x}_1 - \dot{x}_2\right) + C_4\left(\dot{x}_2 - \dot{x}_4\right) - \copyright_2\left(x_1(t) - x_2(t)\right) + \copyright_4\left(x_2(t) - x_4(t)\right) = F_2(t)$$

$$m_3\ddot{x}_3 - C_3\left(\dot{x}_1 - \dot{x}_3\right) + C_5\left(\dot{x}_3 - \dot{x}_4\right) - \copyright_3\left(x_1(t) - x_3(t)\right) + \copyright_5\left(x_3(t) - x_4(t)\right) = F_3(t)$$

$$m_4\ddot{x}_4 - C_6\dot{x}_4 + C_4\left(\dot{x}_4 - \dot{x}_2\right) + C_5\left(\dot{x}_4 - \dot{x}_3\right) + \copyright_6\, x_5(t) + \copyright_4\left(x_4(t) - x_2(t)\right)$$
$$+ \copyright_5\left(x_4(t) - x_3(t)\right) = F_4(t)$$

The mass matrix, *damping matrix* and *stiffness matrix*, respectively, become

$$[\mathbf{M}] = \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix} \qquad (0\text{-}4)$$

$$[C] = \begin{bmatrix} C_1 + C_2 + C_3 & -C_2 & -C_3 & 0 \\ -C_2 & C_2 + C_4 & 0 & -C_4 \\ -C_3 & 0 & C_3 + C_5 & -C_5 \\ 0 & -C_4 & -C_5 & C_4 + C_5 + \end{bmatrix} \qquad (0\text{-}5)$$

$$[\mathbf{K}] = \begin{bmatrix} \copyright_1 + \copyright_2 + \copyright_3 & -\copyright_2 & -\copyright_3 & 0 \\ -\copyright_2 & \copyright_2 + \copyright_4 & 0 & -\copyright_4 \\ -\copyright_3 & 0 & \copyright_3 + \copyright_5 & -\copyright_5 \\ 0 & -\copyright_4 & -\copyright_5 & \copyright_4 + \copyright_5 + \copyright_6 \end{bmatrix} \qquad (0\text{-}6)$$

The general principle for generating these matrices, for systems in which the directions of forces and velocities are defined as in figures 1-5 and 1-6, can be summarized in the following way:

(*i*) the mass matrix is diagonal.

(*ii*) a diagonal element in the stiffness or damping matrix is the sum of the spring rates or damping coefficients, respectively, of all springs / dampers connected to the mass indicated by the row number of the element.

(*iii*) an off-diagonal element at a specific row and column position in the stiffness or damping matrix has the opposite (negative) of the value of the spring rate or damping coefficient, respectively, for the connection between the mass indicated by the row number and that indicated by the column number.

Source:
http://nptel.ac.in/courses/112107088/4