Modeling Parametric Vibration of Multistage Gear Systems as a Tool for Design Optimization

James Kuria, John Kihiu

Abstract—This work presents a numerical model developed to simulate the dynamics and vibrations of a multistage tractor gearbox. The effect of time varying mesh stiffness, time varying frictional torque on the gear teeth, lateral and torsional flexibility of the shafts and flexibility of the bearings were included in the model. The model was developed by using the Lagrangian method, and it was applied to study the effect of three design variables on the vibration and stress levels on the gears. The first design variable, module, had little effect on the vibration levels but a higher module resulted to higher bending stress levels. The second design variable, pressure angle, had little effect on the vibration levels, but had a strong effect on the stress levels on the pinion of a high reduction ratio gear pair. A pressure angle of 25^o resulted to lower stress levels for a pinion with 14 teeth than a pressure angle of 20^o. The third design variable, contact ratio, had a very strong effect on both the vibration levels and bending stress levels. Increasing the contact ratio to 2.0 reduced both the vibration levels and bending stress levels significantly. For the gear train design used in this study, a module of 2.5 and contact ratio of 2.0 for the various meshes was found to yield the best combination of low vibration levels and low bending stresses. The model can therefore be used as a tool for obtaining the optimum gear design parameters for a given multistage spur gear train.

Keywords—bending stress levels, frictional torque, gear design parameters, mesh stiffness, multistage gear train, vibration levels.

NOMENCLATURE

$C_{gi}(t)$	Time varying gear mesh damping coefficient
C_{si}	Shaft damping coefficient
h	Time interval
J_i	Mass moment of inertia of rotor i
K	Stiffness matrix
$K_{gi}(t)$	Time varying gear mesh stiffness
T_1	Input torque
T_2, T_3	Output torque
T_p	Number of teeth on pinion
T	Total kinetic energy of the system.
U	Change in potential energy of a system with respect to its potential energy in the static equilibrium position.
W_{di}	Dynamic load
δ_i	Gear teeth relative displacement
$\dot{\delta}_i$	Gear teeth relative velocity
τ_j	Time period for gear pair j
θ_i	Torsional displacement of rotor i
$\dot{\theta}_i$	Torsional velocity of rotor i

I. INTRODUCTION

J. Kuria is with the Department of Mechanical Engineering, Jomo Kenyatta University of Agriculture & Technology, Nairobi, Kenya e-mail: jkuria@eng.jkuat.ac.ke

J. Kihiu is Associate Professor, Department of Mechanical Engineering, Jomo Kenyatta University of Agriculture and Technology, email: kihiusan@yahoo.com

GEARS are essential components in most power transmission applications, such as automobiles, industrial equipment, airplanes, helicopters and marine vessels. These power transmission systems are often operated under high rotational speeds and/ or high torques and hence their dynamic analysis becomes a relevant issue. The dynamic behavior of gear systems is important for two reasons [1]:

i. durability of the gears
ii. vibration and noise

The physical mechanism of gear meshing has a wide spectrum of dynamic characteristics including time varying mesh stiffness and damping changes during meshing cycle [1]. Additionally, the instantaneous number of teeth in contact, governs the load distribution and sliding resistance acting on the individual teeth. These complexities of the gear meshing mechanism have led prior researchers [2]–[8] to adopt analytical or numerical approaches to analyze the dynamic response of a single pair of gears in mesh.

A large number of parameters are involved in the design of a gear system and for this reason, modeling becomes instrumental to understanding the complex behavior of the system. Provided all the key effects are included and the right assumptions made, a dynamic model will be able to simulate the experimental observations and hence the physical system considered. Thus a dynamic model can be used to reduce the need to perform expensive experiments involved in studying similar systems. The models can also be used as efficient design tools to arrive at an optimal configuration for the system in a cost effective manner.

Mechanical power transmission systems are often subjected to static or periodic torsional loading that necessitates the analysis of torsional characteristics of the system [9]. For instance, the drive train of a typical tractor is subjected to periodically varying torque. This torque variation occurs due to, among other reasons, the fluctuating nature of the combustion engine that supplies power to the gearbox [9]. If the frequency of the engine torque variation matches one of the resonant frequencies of the drive train system, large torsional deflections and internal shear stresses occur. Continued operation of the gearbox under such a condition leads to early fatigue failure of the system components [9]. Dynamic analysis of gears is essential for the reduction of noise and vibrations in automobiles, helicopters, machines and other power transmission systems. Sensitivity of the natural frequencies and vibration modes to system parameters provide important information for tuning the natural frequencies away from operating speeds, minimizing response and optimizing structural design [10].

Few models for the dynamic analysis of a multistage gear train have been developed [11]–[14] and those that exist treat either the shafts of the gear system or the gear teeth as rigid bodies depending on the purpose of the analysis. Effect of varying gear design parameters on the dynamics of a multistage gearbox in order to obtain the optimum parameters for a given gear train has also not been explored. Herbert and Daniel [15] showed that gearboxes must be evaluated for dynamic response on an individual basis. There is therefore the need to develop a general model for a multistage gear train vibrations and one that can be used to obtain the optimum gear design parameters (module, addendum and pressure angle) based on vibration levels, dynamic load and dynamic root stress.

With the advancement of Computer Aided Drafting and Design (CADD) softwares like Mechanical Desktop and Inventor series, the design of gear trains in terms of relative sizes has been made easy. With Autodesk inventor, it is possible to simulate the relative movement of various parts in the design and any interference can be corrected at this stage of the design without having to first fabricate the prototype. However, it is necessary to carry out vibration and dynamic analysis in order to predict the performance of the system before the various parts are fabricated. The effect of the various gear design parameters on the vibration and dynamic characteristics also need to be analyzed in order to optimize the design. The aim of this work is to develop a general model to analyze the vibrations of a multistage gear train taking into account time varying mesh stiffness, time varying frictional torque and shaft torsional stiffness. The model will then be used to analyze the effect of gear design parameters on the vibration levels and gear tooth root stress with the aim of identifying the optimum configurations of the gearbox.

II. Methodology

The model developed here is based on a four-stage reduction gearbox (figure 1) with an overall reduction ratio of 54:1. The gearbox contains five pairs of gears in mesh, the input and output inertias, five shafts and bearings. The major assumptions on which the dynamic model is based are as follows:

i. Gears are modeled as rigid disk with radius equal to the base circle radius and flexibility at the gear teeth.
ii. Each gear is supported by a pair of lateral springs to represent the lateral deflection of shafts and bearings.This implies the simplifying assumption that the gear may move laterally but do not tilt.
iii. Shaft torsion is represented by equivalent torsion spring constants.
iv. The casing is assumed to be rigid (deflections are much smaller than the deflections of the gear teeth, shafts and bearings and can be neglected.)
v. Static transmission error effects are much smaller than the dynamic transmission error effects and so they can be neglected [11].
vi. Gear teeth are assumed to be perfectly involute and manufacturing and assembly errors are ignored.
vii. Backlash is not considered in this model. This is because while running at steady state, the gears are loaded in a single direction only and thus tooth separation is not considered.

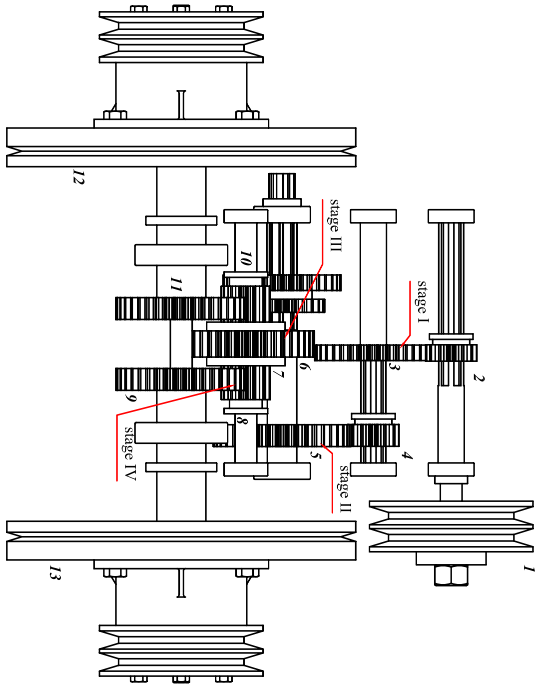

Fig. 1. Gear train for bottom gear ratio

The resulting model is shown in figure 2, while a detailed gear pair model is shown in figure 3. The mathematical model shown in figure 2 can be described by a total of 33 coordinates. The rotational position of the gears, input and output inertias require thirteen coordinates. The lateral positions of the gears due to the lateral deflection of the shafts and bearings require another twenty coordinates.

A set of governing equations of motion for the model was derived using the standard Lagrangian equation, which is given here without proof [16]:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i, \tag{1}$$

Where,

q_i generalized coordinate.
T Total kinetic energy of the system.
U Change in potential energy of a system with respect to its potential energy in the static equilibrium position.
Q_i generalized non-potential forces or moments resulting from excitation forces or moments that add energy into the system, and damping forces and moments that remove energy from it.

The kinetic energy of the system is given by:

$$T = \frac{1}{2}\Sigma\Big(J_i\dot{\theta}_i^2 + m_i\dot{x}_i + m_i\dot{y}_i\Big). \tag{2}$$

The generalized non-potential forces or moments (Q_i) result from excitation forces or moments that add energy into the

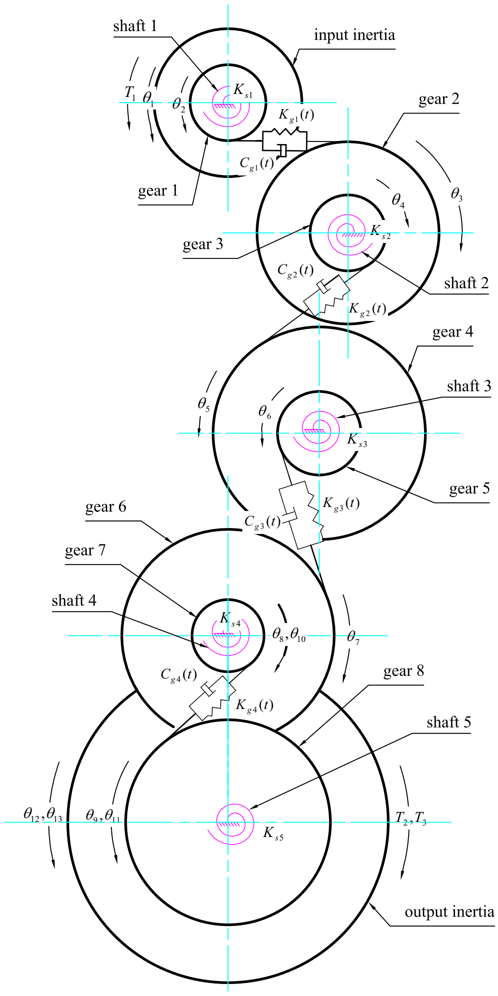

Fig. 2. Gear train model.

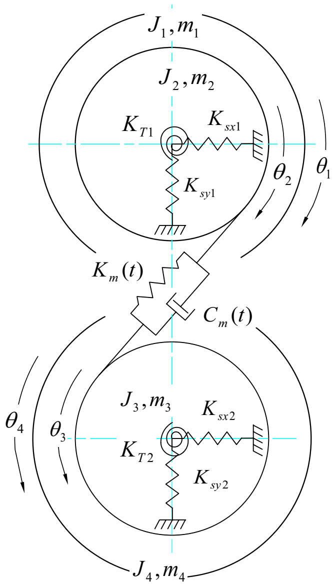

Fig. 3. Detailed gear pair model.

system, and damping forces and moments that remove energy from it. The potential energy is classified into three groups of stored energy caused by:

1) Distortion of the gear meshes, for example the potential energy stored in gear mesh in figure 3 is expressed as:

$$V_{m1} = \frac{1}{2}K_g(t)[R_2\theta_2 - R_3\theta_3 - (y_1 - y_2)cos\gamma + (x_1 - x_2)sin\gamma]^2, \tag{3}$$

2) Twisting of gear shafts, for example the potential energy stored in shaft 1 is expressed as:

$$V_{s1} = \frac{1}{2}K_{s1}[\theta_1 - \theta_2]^2, \tag{4}$$

3) Lateral deflection of the shafts and bearings, expressed as:

$$V_{sl} = \frac{1}{2}\Sigma\Big(K_{xi}x_i^2 + K_{yi}y_i^2\Big). \tag{5}$$

The lateral stiffness of the shafts was obtained by considering the shafts as simply supported. The influence coefficients were then obtained and using the relation $K = A^{-1}$ the stiffness coefficients were obtained [16]. This method ensures that the shaft is statically determinate. The bearing stiffness was obtained by using methods developed by Holm-Hansen and Gao [**?**]. The effective lateral stiffness was then obtained by series combination of the shaft stiffness and bearing stiffness. Under these conditions, the system is described by 33 equations of motion. The sources of forced vibration for the system used in this study were the time varying mesh stiffness and the time varying frictional torque.

The time varying mesh stiffness was obtained by considering the gear tooth as a cantilever beam [17]. The effect of axial deformation, shear deformation and Hertzian contact deformation [18] were included in the stiffness model.

The time varying frictional torque on the gear teeth occurs due to the sliding action of the gear teeth. Sliding friction on the gear tooth surface causes frictional force F_t along the off-line of action direction and a frictional torque T_f about the gear axis. During gear meshing action, the tooth contact point moves along the line of action, and T_f changes continuously due to linearly varying values of the radius of curvature.

The magnitude of the frictional torque is directly related to the friction coefficient and the normal tooth load. Therefore, accurate determination of the friction coefficient is required. In this study, an empirical formula developed by Xu and Kahraman [19] was adopted as it was found to accurately model the instantaneous coefficient of friction along the path of contact of a pair of gears in mesh.

A numerical computer program in FORTRAN code was developed to study the time domain behavior of the system [**?**]. The time domain behavior of the system was obtained by integrating the set of governing differential equations using **4$^{\text{th}}$ order Runge-Kutta** method. The differential equations were linearized by dividing the mesh period of the output pair into many small intervals. The mesh period for any pair of teeth in mesh was taken as the time interval from the initial point of contact to the highest point of single tooth pair contact.

To integrate initial value problems, an appropriate set of initial conditions is required. In this study, all generalized coordinates were set to zero. Starting with this initial estimates of $\theta_i(0)$ and $\dot{\theta}_i(0)$ at the initial contact point, the values of $\theta_i(t)$ and $\dot{\theta}_i(t)$ were calculated for one mesh period of the output pair of gears.

The calculated value of the relative displacement $\delta_i(\tau_j)$ and relative velocity $\dot{\delta}_i(\tau_j)$ after the end of one period τ_j of each pair of gears j in mesh were compared with the initial values $\delta_i(0)$ and $\dot{\delta}_i(0)$. Unless the difference between them was sufficiently small ($\leq 0.002\%$), an iteration procedure was used to obtain the $(i+1)^{th}$ iteration values of $\theta_i(t)$ and $\dot{\theta}_i(t)$ by taking the i^{th} iteration values of $\theta_i(\tau_j)$ and $\dot{\theta}_i(\tau_j)$ as the new initial trial conditions. Once the solution has converged, this state corresponds to the steady state rotational speed of the shafts.

III. Results and Discussion

This section presents the results of the time domain, frequency spectrum, dynamic load and dynamic stress. Table I shows the operating conditions and gear parameters. The

TABLE I
OPERATING CONDITIONS AND GEAR PARAMETERS FOR GEAR TRAIN 1

Input speed	1500 rpm
Nominal Torque	1300 Nm
Module (m)	3.0 mm
Pressure angle	20°
ζ_g	0.1
ζ_s	0.05

relative dynamic displacement of gear i and $i+1$ represents the deflection of the gear teeth from their mean position. If gear i is the driving gear, the following situations will occur [17]:

i. $\delta_i > 0$ This represents the normal operation case and the dynamic mesh force is given by:

$$W_{di} = K_{gi}(t)\delta_i + C_{gi}\dot{\delta}_i, \qquad (6)$$

ii. $\delta_i \leq 0$ and $|\delta_i| \leq bh$,
where bh is the backlash between the gears.
In this case, gears will separate and contact between the gear teeth will be lost.

$$W_{di} = 0, \qquad (7)$$

iii. $\delta_i < 0$ and $bh < |\delta_i|$
In this case, gear $i+1$ will collide with gear i on the back side, and the mesh force will be given by:

$$W_{di} = K_{gi}(t)(\delta_i - bh) + C_{gi}\dot{\delta}_i. \qquad (8)$$

where, W_{di} is the dynamic load. In this study, one of the assumptions in the development of the model was that there was no backlash, therefore only the first case was considered. Figures 4 and 5 show the dynamic transmission error in the time domain and frequency domain for the first two gear meshes. The frequency analysis of the Dynamic Transmission Error (DTE), the relative displacement between the gear teeth, was performed by taking the Fast Fourier Transform (FFT) of its time wave.

In all the time plots, figures 4(a) and 5(a), the tooth cycle is clearly visible with the two distinct regions corresponding to single and double tooth contact. The response is periodic with a period equal to the mesh period τ (shown explicitly in figure 4(a)) as the fundamental meshing period. Larger displacements are seen to occur at the single tooth contact zone due to the lower mesh stiffness in this region.

The effect of reversal of the frictional torque at the pitch point can be seen on figure 4(a), point P. The effect of friction is visible on stages I and II where the rotational speeds are higher. However, at very low speeds, the effect of frictional torque are minimal.

Referring to figure 4(b) and 5(b), it can be seen that the dynamic response corresponds proportionately to the tooth mesh frequency which is the product of the shaft speed and the the number of teeth on the gear. For a perfect tooth, the peak amplitude of the DTE is found at the mesh frequency. The amplitudes of higher harmonics are relatively small and their contribution can be neglected. Both time and frequency spectra indicate that parametric excitations have significant effect on the system response.

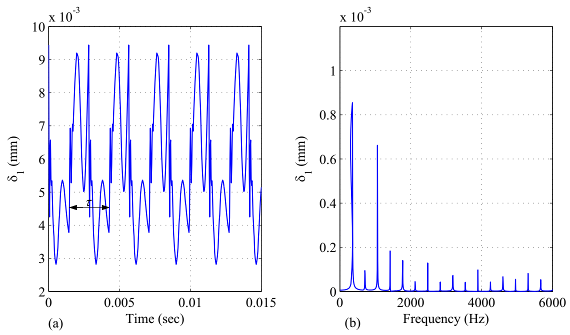

Fig. 4. Vibration signatures for gears in stage I (gear train 1)

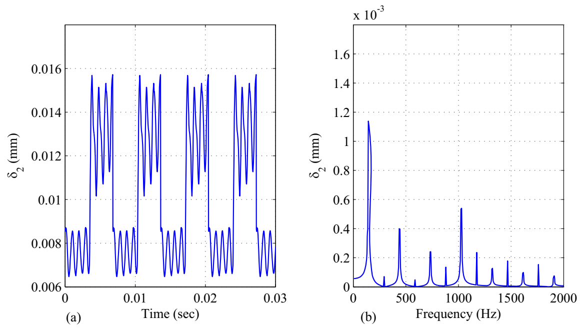

Fig. 5. Vibration signatures for gears in stage II (gear train 1)

The dynamic relationship between all the gear stages are coupled through the non-linear interactions in the gear mesh. The gear mesh forces and moments were evaluated as functions of relative motion and rotation between two meshing gears and the corresponding mesh stiffness as shown in equation 6.

Figure 6 shows the static load and the dynamic load response for a single tooth in mesh for all the reduction stages. The dynamic load is basically a static load sharing in phase with the stiffness change due to the change in the number of teeth in contact superimposed by an oscillating load.

The peak tooth force under dynamic conditions is much higher than the static load especially in the single pair contact region as can be seen in figure 6. Thus if the gear teeth are designed using the static load, there are high possibilities of tooth failure due to the resulting high bending and contact stresses. The dynamic load is also influenced by the pitch-line velocity as shown on table II. The percentage difference between the peak dynamic load and static load decreases as the pitch line velocity reduces.

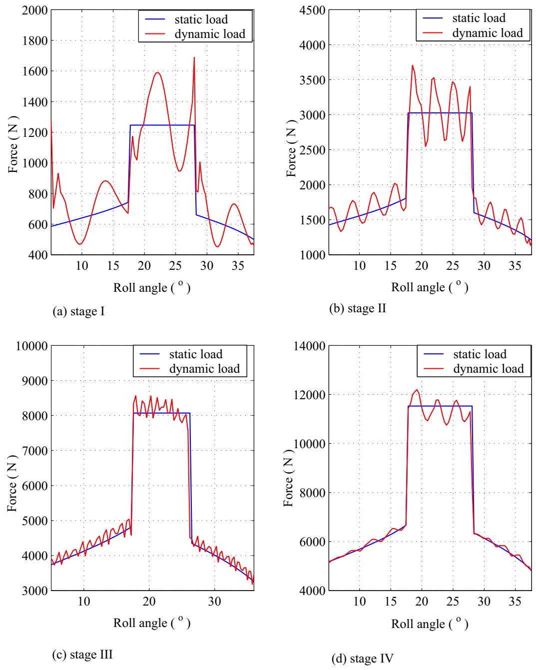

Fig. 6. Comparison of the dynamic and static load on a single tooth over the path of contact (gear train 1)

TABLE II
PERCENTAGE DIFFERENCE BETWEEN PEAK DYNAMIC LOAD AND MAXIMUM STATIC LOAD FOR VARIOUS GEAR MESHES

Gear mesh	Pitch line velocity (m/s)	Static load (N)	Dynamic load (N)	% difference
I	3.30	1246	1611	29
II	1.36	3026	3728	23
III	0.51	8070	8586	6
IV	0.18	11529	12120	5

Figures 7 to 8 compare the static and dynamic stress on a single tooth of all the gears in mesh. The root bending stress on the gear teeth depends on the magnitude of the dynamic force and the position of the force along the path of contact. For the driving gear, the point of contact moves from the lowest point of contact along the tooth profile to the highest point of contact and thus the cantilever beam length of the gear tooth increases along the path of contact. This explains why both the static and dynamic stresses increase with time for the driving gear. The converse is true for the driven gear.

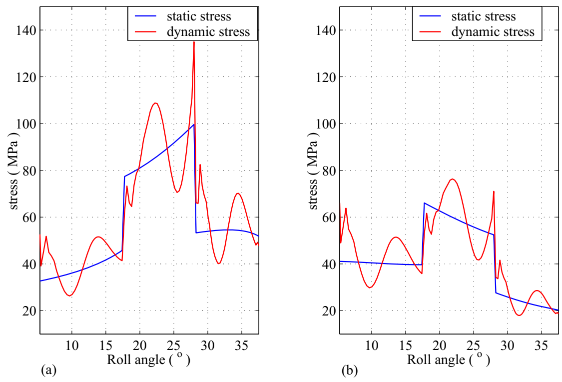

Fig. 7. Tooth bending stress as a function of the contact position for gears in stage I (gear train 1), (a) pinion, (b) gear

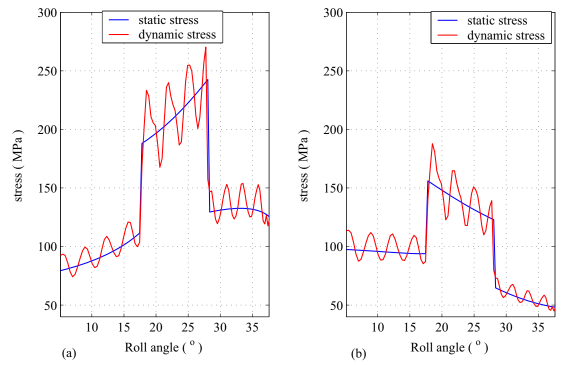

Fig. 8. Tooth bending stress as a function of the contact position for gears in stage II (gear train 1), (a) pinion, (b) gear

A. *Effect of Gear Design Parameters*

In order to optimize the design with respect to gear design parameters, the effect of varying the following gear design parameters was investigated:

i. module
ii. pressure angle
iii. contact ratio

1) Effect of module: The effect of the module of the gear dynamics was investigated by changing the module from 3.0 to 2.5 and 2.0, while holding the pressure angle and pitch radius constant. In order to maintain the pitch radius constant, the number of teeth was varied. However, since the number of teeth for any gear is an integer, the pitch radius of some gears varied slightly (by less than 0.5 mm).

Figure 9 shows the vibration levels of the first two meshes on the gear train in the frequency domain for modules, 2.0, 2.5

and 3.0. It can be seen that the peak amplitude of the vibrations for the different modules are almost the same though gears with module 2.0 show slightly lower amplitudes for mesh I, which resulted from the high mesh stiffness for gears with a smaller module. Figure 10 shows sample dynamic bending stress curves for a pair or gears as function of the contact position. It can be observed that reducing the module of a pair of gears increases the dynamic bending stress significantly. This could be attributed to the smaller tooth thickness at the root for gears with a smaller module.

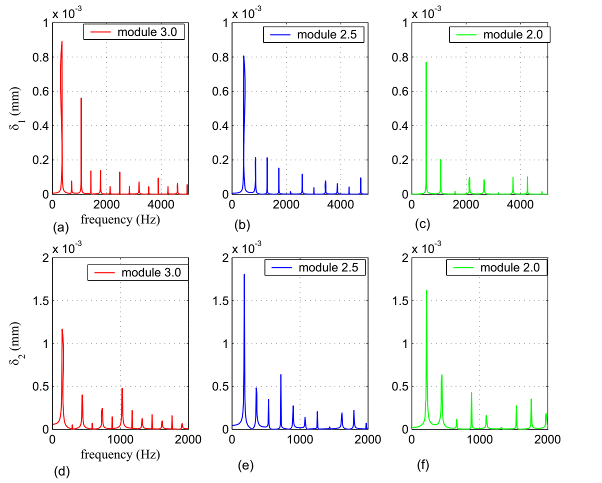

Fig. 9. Vibration levels for various modules for bottom gear ratio (stage I and II)

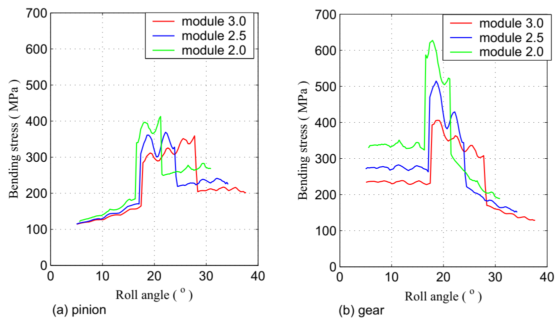

Fig. 10. Root stress on stage IV of gear train 1 for different modules.

2) Effect of Pressure Angle: The pressure angle was increased from 20^o to 25^o while holding the module and number of teeth for the various meshes constant. Figure 11 shows the vibration levels of the first and second meshes. It can be observed that the variation of the vibration levels between gears with a pressure angle of 20^o and 25^o is small and the difference varies from mesh to mesh. However, from the sample bending stress levels (Figure 12) the peak bending stress on the pinion (14T) with a pressure angle of 20^o is higher than that with a pressure angle of 25^o which is attributed to the addendum modification of the teeth with a pressure angle of 20^o to reduce interference. This modification increases the length of the tooth and consequently the cantilever effects on the tooth.

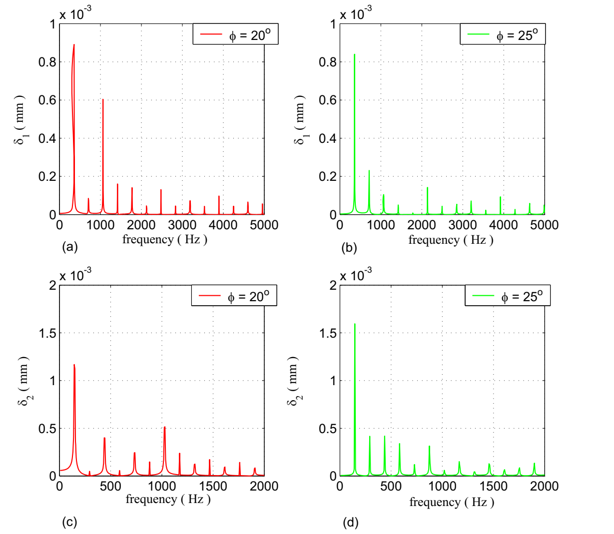

Fig. 11. Comparison of the vibration amplitudes for different pressure angles (stage I and II)

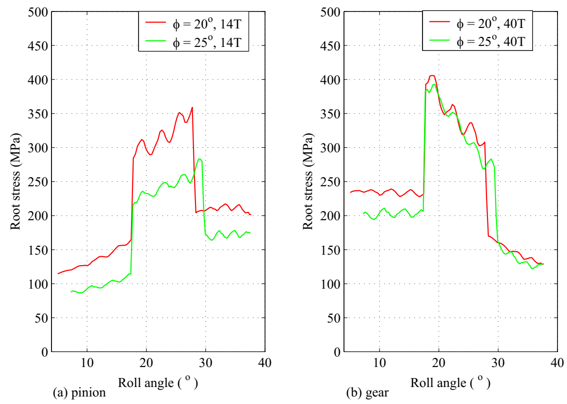

Fig. 12. Sample root stress for gears with different pressure angle.

3) Effect of Contact Ratio: The contact ratio of a pair of gears in mesh is given by equation 9 and is affected by the following parameters:

- addendum
- center distance
- pressure angle
- module

$$C.R = \frac{\sqrt{R_{o1}^2 - R_{b1}^2} + \sqrt{R_{o2}^2 - R_{b2}^2} - (R_{p1} + R_{p2})sin\phi}{p_c cos\phi}. \quad (9)$$

The contact ratio of a gear pair can be increased by varying one of the above parameters or a combination of two or more of these parameters. Increasing the addendum is normally recommended for increasing the contact ratio since this can be achieved by simply adjusting the cutter depth [17]. The maximum permissible addendum modification coefficients are obtained by iteratively varying the addendum modification coefficient of the pinion and gear until the top land thickness is equal to the minimum allowable (usually $0.3m$) [20]. In this research work, a code was developed to obtain the maximum possible contact ratio for a gear pair by varying the addendum and adjusting the center distance in order to avoid interference [?]. Figures 13 and 14 show sample plots for vibration levels of gear pairs with high contact ratio. A pair of gears with a contact ratio close to 2.0 shows relatively low vibration levels especially those with a module of 2.0. A contact ratio of 2.0 reduces the vibration levels by up to 75% in both cases. This effect is due to the very narrow band of single-tooth contact being passed so quickly during gear rotation that the system could not respond until after excitation has passed resulting to a very gentle dynamic response. A contact ratio close to 2.0

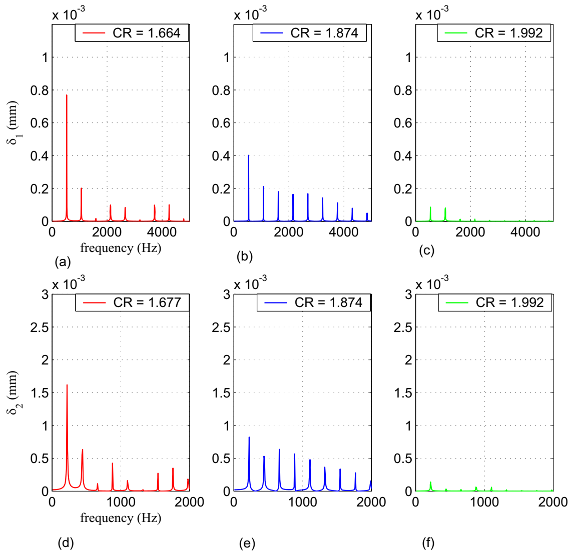

Fig. 13. Sample vibration levels for gear pairs with increased contact ratio (gears with a module of 2.0 mm)

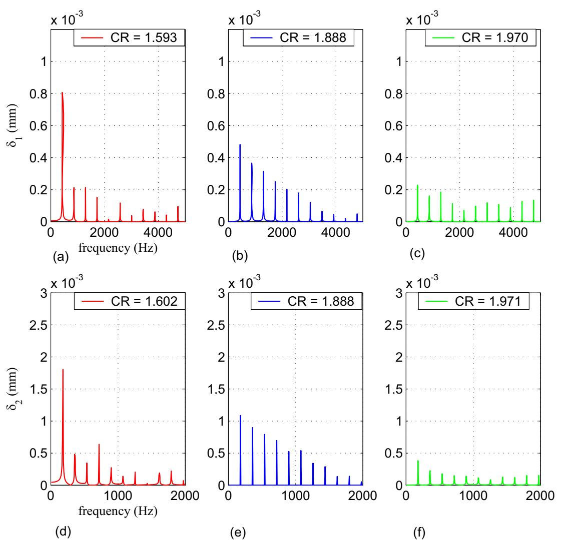

Fig. 14. Sample vibration levels for gear pairs with increased contact ratio (module 2.5 mm)

also results to a smooth root stress curve as shown on figure 15 and 16. A contact ratio of 2.0 reduces the peak dynamic root stresses on the gear teeth by about 45% in both cases. In addition, the discontinuities in the stress curves that occur during the transition from double tooth contact to single tooth contact and vice versa are eliminated. This implies that the gears with a contact ratio of 2.0 would have a higher fatigue life than those with a contact ratio lower than 2.0. However, the gears with a module of 2.5 show lower root stresses than those with a module of 2.0 as shown in figure 17. This could be attributed to the larger tooth thickness on the gears with a module of 2.5. The root stress is dependent on the tooth thickness at the fillet area and the length between the contact point and the critical section of the tooth in addition to the load. The zone of single contact is also eliminated since a contact ratio of 2.0 implies that there are at least 2.0 pairs of teeth in contact at any one point. The only variation in the root bending stress is due to the change in the contact positions.

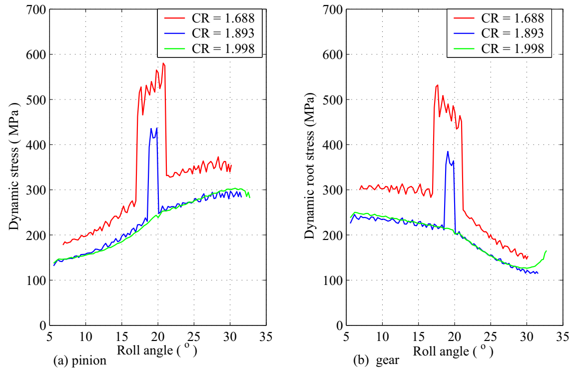

Fig. 15. Root stress on stage IV gears of gear train 1 for different contact ratios using a module of 2.0.

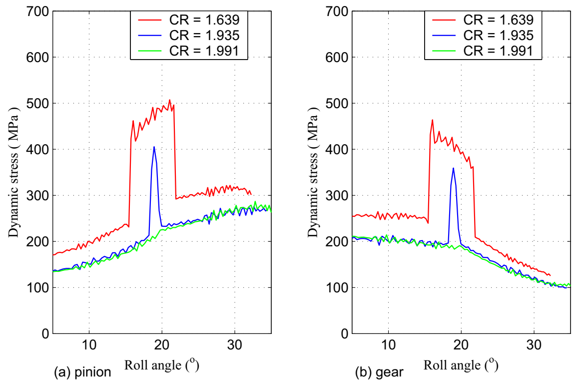

Fig. 16. Root stress on stage IV gears of gear train 1 for different contact ratios using a module of 2.5.

The speed of the gearbox is varied by sliding the speed gears into mesh. This means that the rate of wear for these gears is very high. Thus despite the fact that gears with a module of 2.0 and a contact ratio of 2.0 show lower vibration levels than those with a module of 2.5 and a contact ratio of 2.0, those with a module of 2.5 exhibit lower stresses and have a larger tooth thickness and would therefore be more suitable for this application.

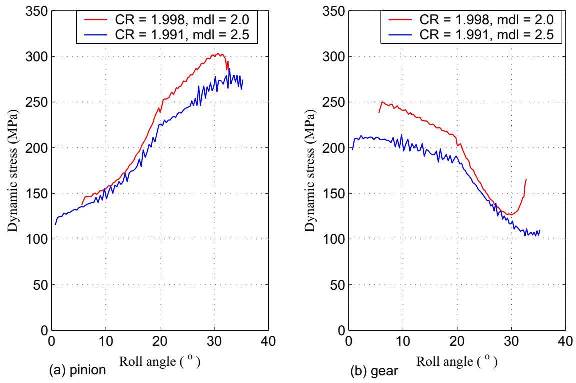

Fig. 17. Root stress on stage IV gears of gear train 1 for a contact ratio of 2.0.

IV. Conclusion

A mathematical model was developed to analyze the dynamic and vibration characteristics of multistage gear transmissions. The model consists of 33 equations of motion which were developed using the Lagrangian method and solved using **Fourth Order Runge Kutta method**. The main sources of excitation for the gear train were the time varying gear mesh stiffness and the time varying frictional torque on the gear teeth. The effect of torsional stiffness of the shafts and lateral stiffness of the shafts and bearing stiffness were considered in the model. Parametric studies were also conducted to examine the effects of three design variables, module, pressure angle and contact ratio. The following following specific conclusions can be drawn from the study:

1) Reducing the module from 3.0 to 2.5 and 2.0 does not yield significant difference in the vibration levels. However, reducing the module increases the dynamic bending stress on the gear teeth and therefore increase gear life. Increased gear life means: Low maintenance costs, low operating costs, increased production of plant/ machine since there will be no downtime, fewer accidents in the plant.
2) Increasing the pressure angle of the gears from 20^o to 25^o has no significant difference on the vibration levels. For gears with a high reduction ratio, increasing the pressure angle results to reduced stress levels since the gears with a higher pressure angle requires less or no addendum modification depending on the number of teeth.
3) Increasing the contact ratio reduces the vibration levels for the various gears in mesh, with the lowest vibration levels recorded for gears with a contact ratio of 2.0. A high contact ratio also leads to lower bending stress levels. Particularly for a contact ratio of 2.0, the discontinuities on the stress curves observed due to variation in the number of teeth in contact is eliminated. Therefore, gears with a contact ratio of 2.0 would have a higher fatigue life. Gears with a module of 2.5 and a contact ratio of 2.0 were found to yield the best combination of low vibration levels and low stress levels and would therefore be recommended for the gear train considered in this research.

References

[1] Tamminana, V. K., Kahraman, A. and Vijayakar, S., "A study of the Relationship Between the dynamic factor and the dynamic transsmission error of spur gear pairs," in *ASME 2005 International Design Engineering Technical Conferences and Computers and Information in Engineering conference*, September 24-28 2005.

[2] M. Vaishya and R. Singh, "Strategies for Modeling Friction in Gear Dynamics," *Journal of Mechanical Design*, vol. 125, pp. 383–393, 2003.

[3] J. H. Kuang and A. D. Lin, "The Effect of Tooth Wear on the Vibration Spectrum of a Spur Gear Pair," *Journal of Vibrations and Acoustics*, vol. 123, pp. 311–317, 2001.

[4] M. Vaishya and R. Singh, "Analysis of Periodically Varying Gear Mesh Systems with Coulomb Friction Using Floquet Theory," *Journal of Sound and Vibration*, vol. 243(3), pp. 525–545, 2001.

[5] R. G. Parker, S. M. Vijayakar and T. Imajo, "Non-linear Dynamic Response of a Spur Gear Pair: Modelling and Experimental Comparisons," *Journal of Sound and Vibrations*, vol. 237(3), pp. 435–455, 2000.

[6] L. Gelman, V. Giurgiutiu and A. Bayoumi, "Statistical Analysis of the Dynamic Mean Excitation for a Spur Gear," *Journal of Vibrations and Acoustics*, vol. 127, pp. 204–207, 2005.

[7] G. Bonori, A. O. Andrisano and F. Pellicano, "Stiffness Evaluation and Vibration in a Tractor Gear," *ASME International Mechanical Engineering Congress and Exposition*, 2004.

[8] M. A. Faith and O. Milosav, "Gear Vibration in Supercritical Mesh-Frequency Range," *Faculty of Mechanical Engineering (FME), Belgrade*, vol. 32, pp. 87–94, 2004.

[9] R. G. Timothy, "Computer- Aided Design Software for Torsional Analysis," Master's thesis, Virginia Polytechnic Institute and State University, 1998.

[10] J. Lin and R. G. Parker, "Sensitivity of Planetory Gear Natural Frequencies and Vibration Modes to Model Parameters," *Journal of Sound and Vibration*, vol. 228(1), pp. 109–128, 1999.

[11] Shengxiang Jia, Ian Howard and Jiande Wang, "The Dynamic Modeling of Multiple Pairs of Spur Gears in Mesh, Including Friction and Geometric Errors," *International Journal of Rotating Machinery*, vol. 9, pp. 437–442, 2003.

[12] T. L. Krantz and M. Rashidi, "Vibration Analysis of a Split Path Gearbox," Tech. Rep., Army Research Laboratory, NASA, 1995.

[13] Jian Lin and R. G. Parker, "Mesh Stiffness Variation Instabilities in a Two stage Gear System," *Transactions of ASME, Journal of Vibration and Acoustics*, vol. 124, pp. 68–76, 2002.

[14] F. K. Choy, Y. K. Tu and D. P. Townsend, "Vibration Signature Analysis of a Multistage Gear Transmission," Tech Rep., NASA Lewis Research Center, 1989.

[15] H. J. Sutherland and D. P. Burwinkle, "The Spectral Content of the Torque Loads on a Turbine Gear Tooth," *Wind Energy, ASME*, vol. 16, pp. 91–97, 1995.

[16] M. L. James, G. M. Smith and P. W. Whaley, *Vibration of Mechanical and Structural systems*. Harper Collins College Publishers, 2nd ed., 1994.

[17] Hsiang-Hsi Lin, *Computer - Aided Design and Analysis of Spur Gear Dynamics*. PhD Thesis, University of Cincinnati, 1985.

[18] K. L. Johnson, *Contact Mechanics*. Cambridge University Press, 1985.

[19] H. Xu and A. Kahraman, "Prediction of Mechanical Efficiency of Parallel - Axis Gear Pairs," *Transactions of ASME*, vol. 129, pp. 58–68, 2007.

[20] D. W. Dudley, *GEAR HANDBOOK: The Design, Manufacture and Application of Gears*. McGraw-Hill Publishing Company, Newyork, 1st ed., 1962.