The Principle of Minimum Potential Energy

The **principle of minimum potential energy** follows directly from the principle of virtual work (for elastic materials).

8.6.1 The Principle of Minimum Potential Energy

Consider again the example given in the last section; in particular re-write Eqn. 8.5.15 as

$$\delta\left\{Pu_B - \left(\frac{E_1A_1}{2L_1} + \frac{E_2A_2}{2L_2}\right)u_B^2\right\} = 0 \tag{8.6.1}$$

The quantity inside the curly brackets is defined to be the **total potential energy** of the system, Π, and the equation states that the variation of Π is zero – that this quantity does not vary when a virtual displacement is imposed:

$$\delta\Pi = 0 \tag{8.6.2}$$

The total potential energy as a function of displacement *u* is sketched in Fig. 8.6.1. With reference to the figure, Eqn. 8.6.2 can be interpreted as follows: the total potential energy attains a stationary value (maximum or minimum) at the *actual* displacement (u_1); for example, $\delta\Pi \neq 0$ for an incorrect displacement u_2. Thus the solution for displacement can be obtained by finding a stationary value of the total potential energy. Indeed, it can be seen that the quantity inside the curly brackets in Fig. 8.6.1 attains a minimum for the solution already derived, Eqn. 8.5.17.

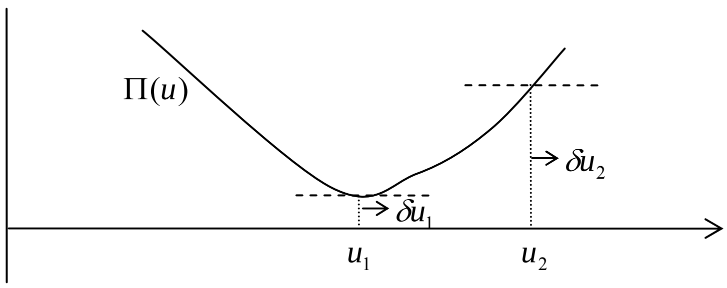

Figure 8.6.1: the total potential energy of a system

To generalise, define the "potential energy" of the applied loads to be $\delta V = -\delta W_{ext}$ so that

$$\delta\Pi = \delta U + \delta V \tag{8.6.3}$$

The external loads must be conservative, precluding for example any sliding frictional loading. Taking the total potential energy to be a function of displacement *u*, one has

$$\delta\Pi = \frac{d\Pi(u)}{du}\delta u = 0 \tag{8.6.4}$$

Thus of all possible displacements *u* satisfying the loading and boundary conditions, the actual displacement is that which gives rise to a stationary point $d\Pi / du = 0$ and the problem reduces to finding a stationary value of the total potential energy $\Pi = U + V$.

Stability

To be precise, Eqn. 8.6.2 only demands that the total potential energy has a stationary point, and in that sense it is called the **principle of stationary potential energy**. One can have a number of stationary points as sketched in Fig. 8.6.2. The true displacement is one of the stationary values u_1, u_2, u_3.

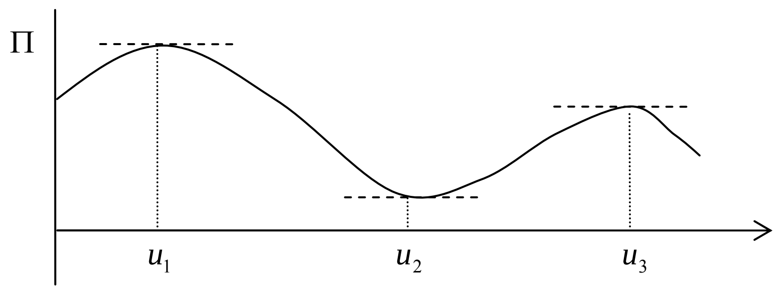

Figure 8.6.2: the total potential energy of a system

Consider the system with displacement u_2. If an external force acts to give the particles of the system some small initial velocity and hence kinetic energy, one has $0 = \Delta\Pi + \Delta K$. The particles will now move and so the displacement u_2 changes. Since Π is a minimum there it must increase and so the kinetic energy must decrease, and so the particles remain close to the equilibrium position. For this reason u_2 is defined as a **stable** equilibrium point of the system. If on the other hand the particles of the body were given small initial velocities from an initial displacement u_1 or u_3, the kinetic energy would increase dramatically; these points are called **unstable** equilibrium points. Only the state of stable equilibrium is of interest here and the principle of stationary potential energy in this case becomes the principle of minimum potential energy.

8.6.2 The Rayleigh-Ritz Method

In applications, the principle of minimum potential energy is used to obtain *approximate* solutions to problems which are otherwise difficult or, more usually, impossible to solve exactly. It forms one basis of the **Finite Element Method** (FEM), a general technique for solving systems of equations which arise in complex solid mechanics problems (and which is discussed in Book III).

Example

Consider a uniaxial bar of length *L*, young's modulus *E* and varying cross-section $A = A_0(1 + x/L)$, fixed at one end and subjected to a force *F* at the other. The true solution for displacement to this problem is $u = (FL/EA_0)\ln(1 + x/L)$. To see how this might be approximated using the principle, one writes

$$\Pi = U + V = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx - Fu\big|_{x=L} \tag{8.6.5}$$

First, substituting in the exact solution leads to

$$\Pi = \frac{EA_0}{2}\int_0^L (1 + x/L)\left(\frac{F}{EA_0}\frac{1}{1 + x/L}\right)^2 dx - F\frac{FL}{EA_0}\ln 2 = -\frac{\ln 2}{2}\frac{F^2 L}{EA_0} \tag{8.6.6}$$

According to the principle, any other displacement solution (which satisfies the displacement boundary condition $u(0) = 0$) will lead to a greater potential energy Π.

Suppose now that the solution was unknown. In that case an estimate of the solution can be made in terms of some unknown parameter(s), substituted into Eqn. 8.6.5, and then minimised to find the parameters. This procedure is known as the **Rayleigh Ritz method**. For example, let the guess, or **trial function**, be the linear function $u = \alpha + \beta x$. The boundary condition leads to $\alpha = 0$. Substituting $u = \beta x$ into Eqn. 8.6.5 leads to

$$\Pi = \frac{1}{2}EA_0\beta^2\int_0^L (1 + x/L)dx - F\beta L = \frac{3}{4}EA_0 L\beta^2 - F\beta L \tag{8.6.7}$$

The principle states that $\delta\Pi = (d\Pi/d\beta)\delta\beta = 0$, so that

$$\frac{d\Pi}{d\beta} = \frac{3}{2}EA_0 L\beta - FL = 0 \quad \to \quad \beta = \frac{2F}{3EA_0} \quad \to \quad u = \frac{2Fx}{3EA_0} \tag{8.6.8}$$

The exact and approximate Ritz solution are plotted in Fig. 8.6.3.

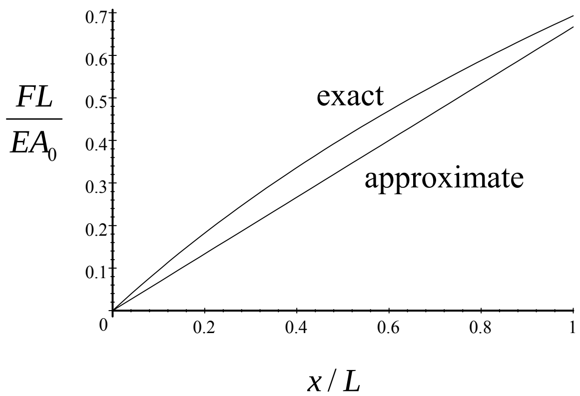

Figure 8.6.3: exact and (Ritz) approximate solution for axial problem

The total potential energy due to this approximate solution $2Fx/3EA_0$ is, from Eqn. 8.6.5,

$$\Pi = -\frac{1}{3}\frac{F^2 L}{EA_0} \tag{8.6.9}$$

which is indeed greater than the minimum value Eqn. 8.6.6 ($\approx -0.347F^2 L/EA_0$).

■

The accuracy of the solution 8.6.9 can be improved by using as the trial function a quadratic instead of a linear one, say $u = \alpha + \beta x + \gamma x^2$. Again the boundary condition leads to $\alpha = 0$. Then $u = \beta x + \gamma x^2$ and there are now two unknowns to determine. Since Π is a function of two variables,

$$\delta\Pi(\beta,\gamma) = \frac{\partial \Pi}{\partial \beta}\delta\beta + \frac{\partial \Pi}{\partial \gamma}\delta\gamma = 0 \tag{8.6.10}$$

and the two unknowns can be obtained from the two conditions

$$\frac{\partial \Pi}{\partial \beta} = 0, \qquad \frac{\partial \Pi}{\partial \gamma} = 0 \tag{8.6.11}$$

Example

A beam of length L and constant Young's modulus E and moment of inertia I is supported at its ends and subjected to a uniform distributed force per length f. Let the beam undergo deflection $v(x)$. The potential energy of the applied loads is

$$V = -\int_0^L f v(x)dx \tag{8.6.12}$$

and, with Eqn. 8.5.19, the total potential energy is

$$\Pi = \frac{EI}{2}\int_0^L \left(\frac{d^2 v}{dx^2}\right)^2 dx - f\int_0^L v dx \tag{8.6.13}$$

Choose a quadratic trial function $v = \alpha + \beta x + \gamma x^2$. The boundary conditions lead to $v = \gamma x(x - L)$. Substituting into 8.6.13 leads to

$$\Pi = 2\gamma^2 EIL - f\gamma L^3/6 \tag{8.6.14}$$

With $\delta\Pi = (d\Pi/d\gamma)\delta\gamma = 0$, one finds that

Source: http://homepages.engineering.auckland.ac.nz/~pkel015/SolidMechanicsBooks/Part_I/BookSM_Part_I/08_Energy/08_Energy_06_Minimum_Potential_Energy.pdf