Members Subjected to Axisymmetric Loads

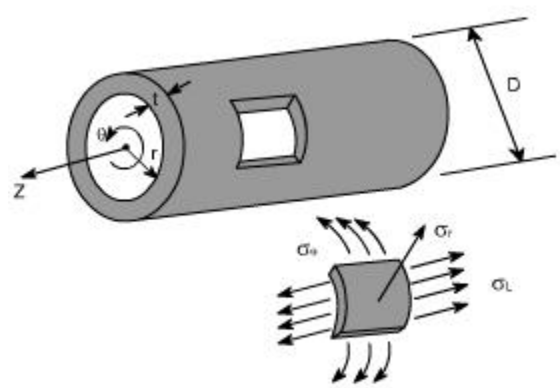
Pressurized thin walled cylinder:

Preamble : Pressure vessels are exceedingly important in industry. Normally two types of pressure vessel are used in common practice such as cylindrical pressure vessel and spherical pressure vessel.

In the analysis of this walled cylinders subjected to internal pressures it is assumed that the radial plans remains radial and the wall thickness dose not change due to internal pressure. Although the internal pressure acting on the wall causes a local compressive stresses (equal to pressure) but its value is neglibly small as compared to other stresses & hence the sate of stress of an element of a thin walled pressure is considered a biaxial one.

Further in the analysis of them walled cylinders, the weight of the fluid is considered neglible.

Let us consider a long cylinder of circular cross - section with an internal radius of R_2 and a constant wall thickness'‘t' as showing fig.



This cylinder is subjected to a difference of hydrostatic pressure of ‘p' between its inner and outer surfaces. In many cases, ‘p' between gage pressure within the cylinder, taking outside pressure to be ambient.

By thin walled cylinder we mean that the thickness‘t' is very much smaller than the radius R_i and we may quantify this by stating than the ratio t / R_i of thickness of radius should be less than 0.1.

An appropriate co-ordinate system to be used to describe such a system is the cylindrical polar one r, θ , z shown, where z axis lies along the axis of the cylinder, r is radial to it and θ is the angular co-ordinate about the axis.

The small piece of the cylinder wall is shown in isolation, and stresses in respective direction have also been shown.

Type of failure:

Such a component fails in since when subjected to an excessively high internal pressure. While it might fail by bursting along a path following the circumference of the cylinder. Under normal circumstance it fails by circumstances it fails by bursting along a path parallel to the axis. This suggests that the hoop stress is significantly higher than the axial stress.

In order to derive the expressions for various stresses we make following

Applications :

Liquid storage tanks and containers, water pipes, boilers, submarine hulls, and certain air plane components are common examples of thin walled cylinders and spheres, roof domes.

ANALYSIS : In order to analyse the thin walled cylinders, let us make the following assumptions :

- There are no shear stresses acting in the wall.
- The longitudinal and hoop stresses do not vary through the wall.
- Radial stresses σ_r which acts normal to the curved plane of the isolated element are neglibly small as compared to other two stresses especially when $\left[\frac{t}{R_i} < \frac{1}{20}\right]$

The state of tress for an element of a thin walled pressure vessel is considered to be biaxial, although the internal pressure acting normal to the wall causes a local compressive stress equal to the internal pressure, Actually a state of tri-axial stress exists on the inside of the vessel. However, for then walled pressure vessel the third stress is much smaller than the other two stresses and for this reason in can be neglected.

Thin Cylinders Subjected to Internal Pressure:

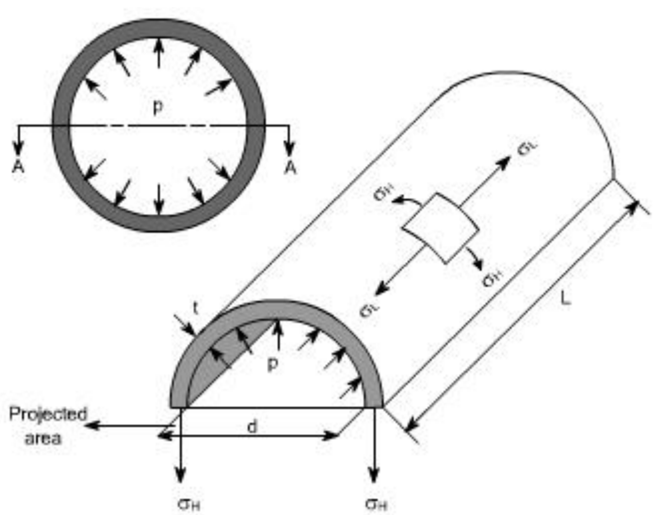
When a thin – walled cylinder is subjected to internal pressure, three mutually perpendicular principal stresses will be set up in the cylinder materials, namely

- Circumferential or hoop stress
- The radial stress
- Longitudinal stress

now let us define these stresses and determine the expressions for them

Hoop or circumferential stress:

This is the stress which is set up in resisting the bursting effect of the applied pressure and can be most conveniently treated by considering the equilibrium of the cylinder.



In the figure we have shown a one half of the cylinder. This cylinder is subjected to an internal pressure p.

i.e. p = internal pressure

d = inside diametre

L = Length of the cylinder

t = thickness of the wall

Total force on one half of the cylinder owing to the internal pressure 'p'

= p x Projected Area

= p x d x L

= **p .d. L** ------- (1)

The total resisting force owing to hoop stresses σ_H set up in the cylinder walls

= **2 .σ_H .L.t** ---------(2)

Because σ□$_H$.L.t. is the force in the one wall of the half cylinder.

the equations (1) & (2) we get

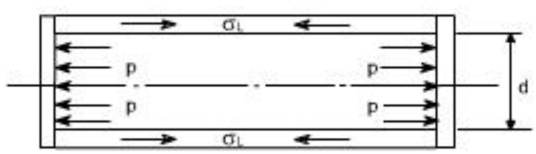
2 . σ_H . L . t = p . d . L

σ_H = (p . d) / 2t

Circumferential or hoop Stress (σ_H) = (p .d)/ 2t

Longitudinal Stress:

Consider now again the same figure and the vessel could be considered to have closed ends and contains a fluid under a gage pressure p.Then the walls of the cylinder will have a longitudinal stress as well as a ciccumferential stress.



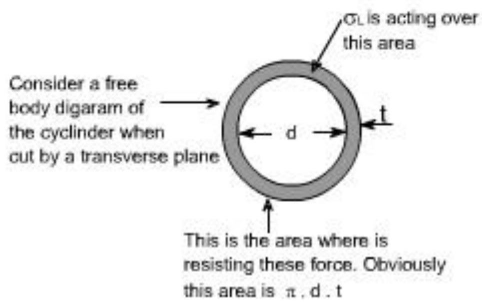
Total force on the end of the cylinder owing to internal pressure

= pressure x area

= p x π□d^2 /4

Area of metal resisting this force = πd.t. (approximately)

because πd is the circumference and this is multiplied by the wall thickness



Hence the longitudnal stresses

$$\text{Set up} = \frac{\text{force}}{\text{area}} = \frac{[p \times \pi d^2/4]}{\pi d t}$$

$$= \frac{pd}{4t} \quad \text{or} \quad \sigma_L = \frac{pd}{4t}$$

or alternatively from equilibrium conditions

$$\sigma_L . (\pi d t) = p . \frac{\pi d^2}{4}$$

$$\text{Thus} \boxed{\sigma_L = \frac{pd}{4t}}$$

Change in Dimensions :

The change in length of the cylinder may be determined from the longitudinal strain.

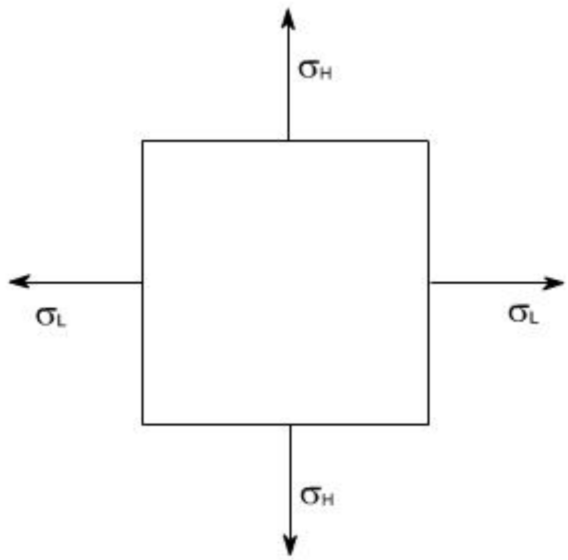
Since whenever the cylinder will elongate in axial direction or longitudinal direction, this will also get decreased in diametre or the lateral strain will also take place. Therefore we will have to also take into consideration the lateral strain.as we know that the poisson's ratio (*v*) is

$$\nu = \frac{-\text{lateral strain}}{\text{longitudnal strain}}$$

where the -ve sign emphasized that the change is negative

Consider an element of cylinder wall which is subjected to two mutually $\perp^r$ normal stresses□σ_L and σ_H .

Let E = Young's modulus of elasticity



Resultant Strain in longitudnal direction $= \frac{\sigma_L}{E} - \nu \frac{\sigma_H}{E} = \frac{1}{E}(\sigma_L - \nu\sigma_H)$

recalling

$$\sigma_L = \frac{pd}{4t} \qquad \sigma_H = \frac{pd}{2t}$$

$\in_1$ (longitudnal strain) $= \frac{pd}{4Et}[1-2\nu]$

or

Change in Length = Longitudalstrain x original Length

$= \in_1 . L$

Similarly the hoop Strain $\in_2 = \frac{1}{E}(\sigma_H - \nu\sigma_L) = \frac{1}{E}\left[\frac{pd}{2t} - \nu\frac{pd}{4t}\right]$

$$\in_2 = \frac{pd}{4Et}[2-\nu]$$

Infact $\in_2$ is the hoop strain if we just go by the definition then

$$\in_2 = \frac{\text{Changein diametre}}{\text{Original diametre}} = \frac{\delta d}{d}$$

where d = original diameter.

if we are interested to find out the change in diametre then

Changein diametre $= \in_2$ .Original diametre

i.e $\delta d = \in_2 .d$ substituting the value of $\in_2$ we get

$$\delta d = \frac{p.d}{4.t.E}[2-\nu].d$$

$$= \frac{p.d^2}{4.t.E}[2-\nu]$$

i.e $\boxed{\delta d = \frac{p.d^2}{4.t.E}[2-\nu]}$

Volumetric Strain or Change in the Internal Volume:

When the thin cylinder is subjected to the internal pressure as we have already calculated that there is a change in the cylinder dimensions i.e, longitudinal strain and hoop strains come into picture. As a result of which there will be change in capacity of the cylinder or there is a change in the volume of the cylinder hence it becomes imperative to determine the change in volume or the volumetric strain.

The capacity of a cylinder is defined as

V = Area X Length

$= \pi d^2/4 \times L$

Let there be a change in dimensions occurs, when the thin cylinder is subjected to an internal pressure.

(i) The diameter **d** changes to $\rightarrow$ **d + δ d**

(ii) The length **L** changes to $\rightarrow$ **L + δ L**

Therefore, the change in volume = Final volume − Original volume

$$= \frac{\pi}{4}[d+\delta d]^2.(L+\delta L) - \frac{\pi}{4}d^2.L$$

$$\text{Volumetric strain} = \frac{\text{Change in volume}}{\text{Original volume}} = \frac{\frac{\pi}{4}[d+\delta d]^2.(L+\delta L) - \frac{\pi}{4}d^2.L}{\frac{\pi}{4}d^2.L}$$

$$\epsilon_v = \frac{\left\{[d+\delta d]^2.(L+\delta L) - d^2.L\right\}}{d^2.L} = \frac{\left\{(d^2+\delta d^2+2d.\delta d).(L+\delta L) - d^2.L\right\}}{d^2.L}$$

simplifying and neglecting the products and squares of small quantities, i.e. δd & δL hence

$$= \frac{2d.\delta d.L + \delta L.d^2}{d^2 L} = \frac{\delta L}{L} + 2.\frac{\delta d}{d}$$

By definition $\frac{\delta L}{L}$ = Longitudnal strain

$\frac{\delta d}{d}$ = hoop strain, Thus

Volumetric strain = longitudnal strain + 2 x hoop strain

on substituting the value of longitudnal and hoop strains we get

$$\epsilon_1 = \frac{pd}{4tE}[1-2\nu] \quad \& \quad \epsilon_2 = \frac{pd}{4tE}[1-2\nu]$$

$$\text{or Volumetric} = \epsilon_1 + 2\epsilon_2 = \frac{pd}{4tE}[1-2\nu] + 2.\left(\frac{pd}{4tE}[1-2\nu]\right)$$

$$= \frac{pd}{4tE}\{1-2\nu+4-2\nu\} = \frac{pd}{4tE}[5-4\nu]$$

$$\text{Volumetric Strain} = \frac{pd}{4tE}[5-4\nu] \quad \text{or} \quad \boxed{\epsilon_v = \frac{pd}{4tE}[5-4\nu]}$$

Therefore to find but the increase in capacity or volume, multiply the volumetric strain by original volume.

Hence

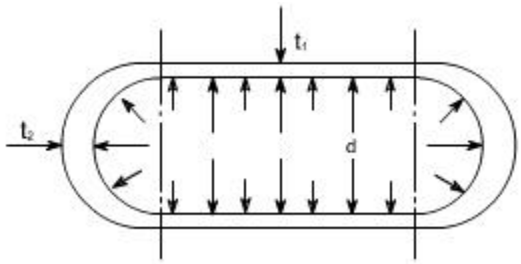
Change in Capacity / Volume or

$$\text{Increase in volume} = \frac{pd}{4tE}[5 - 4\nu]V$$

Cylindrical Vessel with Hemispherical Ends:

Let us now consider the vessel with hemispherical ends. The wall thickness of the cylindrical and hemispherical portion is different. While the internal diameter of both the portions is assumed to be equal

Let the cylindrical vassal is subjected to an internal pressure p.



For the Cylindrical Portion

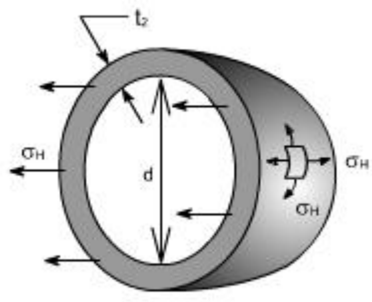
$$\text{hoop or circumferential stress} = \sigma_{HC} = \frac{pd}{2t_1}$$ 'c' here synifies the cylindrical portion.

$$\text{longitudnal stress} = \sigma_{LC} = \frac{pd}{4t_1}$$

$$\text{hoop or circumferential strain } \epsilon_2 = \frac{\sigma_{HC}}{E} - \nu\frac{\sigma_{LC}}{E} = \frac{pd}{4t_1E}[2-\nu]$$

$$\text{or} \quad \epsilon_2 = \frac{pd}{4t_1E}[2-\nu]$$

<u>For The Hemispherical Ends:</u>



Because of the symmetry of the sphere the stresses set up owing to internal pressure will be two mutually perpendicular hoops or circumferential stresses of equal values. Again the radial stresses are neglected in comparison to the hoop stresses as with this cylinder having thickness to diametre less than1:20.

Consider the equilibrium of the half – sphere

Force on half-sphere owing to internal pressure = pressure x projected Area

$= p.\ \pi d^2/4$

$$\text{Resisting force} = \sigma_H \, . \, \pi . d . t_2$$

$$\therefore \qquad p . \frac{\pi . d^2}{4} = \sigma_H \, . \, \pi d . t_2$$

$$\Rightarrow \sigma_H \text{ (for sphere)} = \frac{pd}{4t_2}$$

$$\text{similarly the hoop strain} = \frac{1}{E}\left[\sigma_H - \nu . \sigma_H\right] = \frac{\sigma_H}{E}[1-\nu] = \frac{pd}{4t_2E}[1-\nu] \text{ or } \boxed{\epsilon_{2s} = \frac{\mathbf{pd}}{\mathbf{4t_2E}}[\mathbf{1-\nu}]}$$

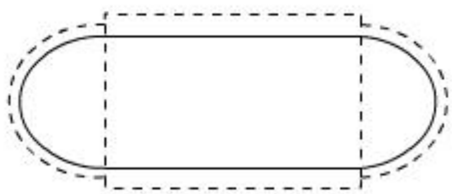


Fig – shown the (by way of dotted lines) the tendency, for the cylindrical portion and the spherical ends to expand by a different amount under the action of internal pressure. So owing to difference in stress, the two portions (i.e. cylindrical and spherical ends) expand by a different amount. This incompatibly of deformations causes a local bending and sheering stresses in the neighborhood of the joint. Since there must be physical continuity between the ends and the cylindrical portion, for this reason, properly curved ends must be used for pressure vessels.

Thus equating the two strains in order that there shall be no distortion of the junction

$$\frac{pd}{4t_1E}[2-\nu] = \frac{pd}{4t_2E}[1-\nu] \text{ or } \frac{t_2}{t_1} = \frac{1-\nu}{2-\nu}$$

But for general steel works ν = 0.3, therefore, the thickness ratios becomes

$\mathbf{t_2 / t_1 = 0.7/1.7}$ or

$$\boxed{\mathbf{t_1 = 2.4\ t_2}}$$

i.e. the thickness of the cylinder walls must be approximately 2.4 times that of the hemispheroid ends for no distortion of the junction to occur.

SUMMARY OF THE RESULTS : Let us summarise the derived results

(A) The stresses set up in the walls of a thin cylinder owing to an internal pressure p are :

(i) Circumferential or loop stress

$\sigma_H = pd/2t$

(ii) Longitudinal or axial stress

$\sigma_L = pd/4t$

Where d is the internal diametre and t is the wall thickness of the cylinder.

then

Longitudinal strain $\in_L$ **$= 1 / E [\sigma_L - \nu \sigma_H]$**

Hoop stain $\in_H$ **$= 1 / E [\sigma_H - \nu \sigma_L]$**

(B) Change of internal volume of cylinder under pressure

$$= \frac{pd}{4tE}[5 - 4\nu]V$$

(C) Fro thin spheres circumferential or loop stress

$$\sigma_H = \frac{pd}{4t}$$

<u>Thin rotating ring or cylinder</u>

Consider a thin ring or cylinder as shown in Fig below subjected to a radial internal pressure p caused by the centrifugal effect of its own mass when rotating. The centrifugal effect on a unit length of the circumference is

$p = m\,\omega^2 r$

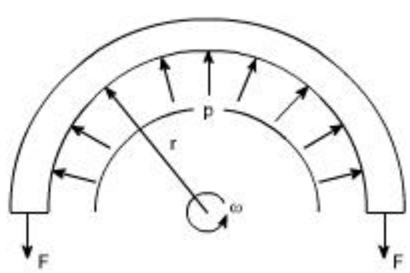


Fig 19.1: Thin ring rotating with constant angular velocity ω

Here the radial pressure 'p' is acting per unit length and is caused by the centrifugal effect if its own mass when rotating.

Thus considering the equilibrium of half the ring shown in the figure,

2F = p x 2r (assuming unit length), as 2r is the projected area

$F = pr$

Where F is the hoop tension set up owing to rotation.

The cylinder wall is assumed to be so thin that the centrifugal effect can be assumed constant across the wall thickness.

F = mass x acceleration = $\mathbf{m\,\omega^2\,r \times r}$

This tension is transmitted through the complete circumference and therefore is resisted by the complete cross – sectional area.

hoop stress = F/A = $\mathbf{m\,\omega^2\,r^2 / A}$

Where A is the cross – sectional area of the ring.

Now with unit length assumed m/A is the mass of the material per unit volume, i.e. the density ρ .

hoop stress = $\boldsymbol{\rho\,\omega^2\,r^2}$

$\boldsymbol{\sigma_H = \rho\,.\,\omega^2\,.\,r^2}$

Source: http://nptel.ac.in/courses/Webcourse-contents/IIT-ROORKEE/strength%20of%20materials/homepage.htm