MATHEMATICAL DERIVATIONS OF THE FOUCAULT PENDULUM

Picture 15. Image
A Wheatstone pendulum setup with the attachment points aligned with the central axis of rotation.

I will start with a derivation for the Wheatstone pendulum setup that is depicted in image 15. That setup is effectively a 2-dimensional case; all forces that are involved act parallel to the plane of the equator; all motion is in a plane that is parallel to the equator. At the very end of the mathematical discussion I will add the modification that generalizes the result to cases where the suspension points are not aligned with the central axis of rotation.

The equation of motion

As discussed in the section Decomposition in vector components the mathematics of the equation of motion is simplified by representing the force exerted upon the pendulum bob as a combination of two forces:

· Centripetal force that sustains the rotation
· Restoring force that sustains the vibration

Both can be treated as a harmonic force.

The centripetal force that sustains rotation with constant angular velocity Ω, for a coordinate system with the zero point at the central axis, is given by:

$$\begin{cases} F_{c,x} = -m\Omega^2 x \\ F_{c,y} = -m\Omega^2 y \end{cases}$$

The restoring force acts towards the equilibrium point (plumb line direction) of the pendulum. The coordinate system can be chosen in such a way that the central axis and the equilibrium point are both on the y-axis. Let the y-coordinate of the equilibrium point be called y_e. Let the frequency of the pendulum swing be called ψ.

$$\begin{cases} F_{r,x} = -m\psi^2 x \\ F_{r,y} = -m\psi^2 (y - y_e) \end{cases}$$

The centrifugal term and the Coriolis term

I assume that the reader is at ease with the centrifugal term and the Coriolis term. The necessary information is present in the articles Rotational-vibrational coupling and Oceanography: inertial oscillations and in the following interactive animation Coriolis effect. In the rotational-vibrational coupling article I present a derivation of the Coriolis term (the centrifugal term is trivial), and in the inertial oscillation article I show how things work out for terrestrial effects.

To help recognize the notation: in the following system of equations (which is for motion relative to a coordinate system that rotates with angular velocity Ω) the term that is proportional to x and y is the centrifugal term. The Coriolis term is proportional to dx/dt and dy/dt respectively.

The acceleration that is associated with the Coriolis effect is perpendicular to the velocity. If you have a vector dx/dy, then the vector perpendicular to that is -dy/dx; it's the negative, and inverted. In the system of motion equations (for x-direction and y-direction) you see that the factor 2Ωdy/dt is in the equation for acceleration in the x-direction, and vice versa.

$$\begin{cases} \dfrac{d^2x}{dt^2} = \Omega^2 x + 2\Omega \dfrac{dy}{dt} \\ \dfrac{d^2y}{dt^2} = \Omega^2 y - 2\Omega \dfrac{dx}{dt} \end{cases}$$

The full equation

No factor m

Let me explain first why I have omitted the factor 'm' for the mass in the equation of motion below. The restoring force arises from elasticity. If the bob is replaced with a heavier bob then the elastic material stretches some more before settling into an equilibrium state. The required force is proportional to m, and the setup simply self-adjusts to provide the required force.

The full equation of motion for motion with respect to a rotating coordinate system has four factors: the two components of the force plus the centrifugal term and the Coriolis term.

centrifugal centripetal restoring Coriolis

$$\begin{cases} \dfrac{d^2x}{dt^2} = \Omega^2 x - \Omega^2 x - \psi^2 x \qquad + 2\Omega\dfrac{dy}{dt} \\ \dfrac{d^2y}{dt^2} = \Omega^2 y - \Omega^2 y - \psi^2 (y - y_e) \ - 2\Omega\dfrac{dx}{dt} \end{cases}$$

It's assured that 'centrifugal' and 'centripetal' drop away against each other because the system is self-adjusting: if the angular velocity of the system would increase then the springs deform a little more until the point is reached where the springs once more provide the required amount of centripetal force.

Letting the centrifugal term and the expression for the centripetal force drop away against each other:

$$\begin{cases} \dfrac{d^2x}{dt^2} = -\psi^2 x \qquad + 2\Omega\dfrac{dy}{dt} \\ \dfrac{d^2y}{dt^2} = -\psi^2 (y - y_e) \ - 2\Omega\dfrac{dx}{dt} \end{cases}$$

This equation of motion describes the effects of the centripetal force on the motion pattern.

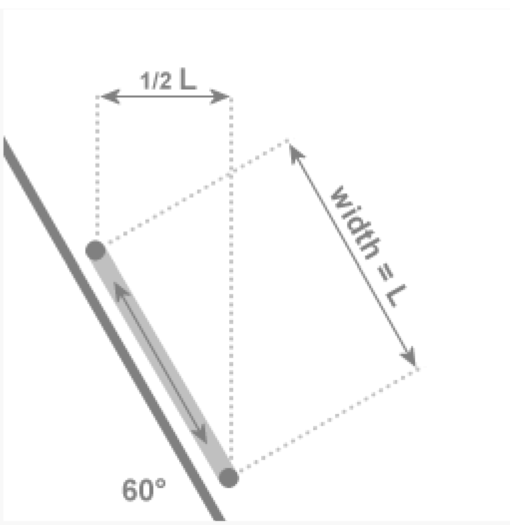
The above expression can be simplified further by shifting the zero point of the coordinate system. The Coriolis term only contains velocity with respect to the rotating system, so it is independent of where the zero point of the coordinate system is positioned. In the following equations *x* and *y* are not the distance to the central axis but the distance to the center point of the vibration.

$$\begin{cases} \frac{d^2x}{dt^2} = -\psi^2 x + 2\Omega\frac{dy}{dt} \\ \frac{d^2y}{dt^2} = -\psi^2 y - 2\Omega\frac{dx}{dt} \end{cases}$$

As a reminder: the above equations apply for only a single case, the case when the suspension of the pendulum is aligned with the central axis of rotation, as depicted in image 15.

This system of equations is the same as the equations of motion for two coupled oscillators. Here the oscillations are vibration in x-direction and vibration in y-direction. The Coriolis term describes that *acceleration* in x-direction is proportional to *velocity* in y-direction, and that acceleration in y-direction is proportional to velocity in x-direction. In other words: the Coriolis term describes the *transfer* of the vibration direction.

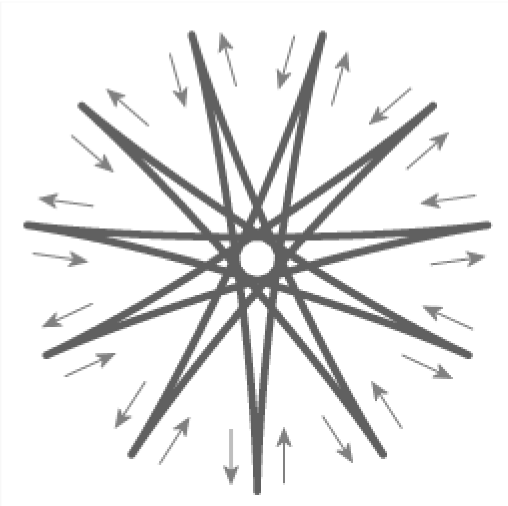
Weakened coupling



Picture 16. Image
In the case of an angle of 60 degrees with the central axis of rotation: If the two extremal points of the swing are a distance of L apart, then the motion towards and away from the central axis covers a distance of 1/2 L

When the attachment points are not aligned with the central axis of rotation the coupling between the rotation and the vibration is weakened. Less motion towards and away from the central axis of rotation means the centripetal force will be doing less work.

$$\begin{cases} \dfrac{d^2x}{dt^2} = -\psi^2 x + 2\Omega \dfrac{dy}{dt} sin(\phi) \\ \dfrac{d^2y}{dt^2} = -\psi^2 y - 2\Omega \dfrac{dx}{dt} sin(\phi) \end{cases}$$



Picture 17. Image
The shape of the pendulum bob's trajectory.

Obtaining an analytic solution to that equation of motion is in thesecond Foucault pendulum article

Image 17 has been created by plotting the analytic solution to the above equation of motion. It represents the case where the ratio of ψ to Ω is 11 to 1 (Usually the ratio of ψ to Ω is in the order of thousands to one). The image depicts the case of releasing the bob in such a way that on release it has no velocity with respect to the rotating system.

What the equations describe

Remarkably, the equations describe that the *shape* of the trajectory of the Foucault pendulum is exactly the same on all latitudes, the latitude of deployment affects only the rate of precession. That means that just from the shape of the trajectory you will not be able to observe the magnitude of the Earth's rotation rate. For instance, if you observe that it takes 32 hours for the pendulum to complete a precession cycle then maybe the Earth takes 32 hours to rotate, or maybe you are on a latitude where the precession takes 32 hours, with the Earth rotating at some unknown faster rate. So if you limit yourself rigidly to using data from the *pendulum motion only* you cannot observe the Earths rotation rate.

The similarity of the shape on every latitude is quite surprising, for in the case of a polar pendulum and in the case of a latitudinal pendulum the mechanism that is involved is entirely different.
In the case of a polar pendulum the only physics taking place is the swing of the pendulum, the Earth merely rotates underneath the pendulum, without affecting it; the "precession" of a polar pendulum is*apparent precession*. On the other hand, in the case of a latitudinal pendulum the pendulum setup as a whole is circumnavigating the Earth's axis and consequently the vibration is affected: there is a coupling of latitudinal and longitudinal vibration. When the coupling is 100% as in the setup depicted in animation 10 the vibration retains the same orientation with respect to inertial space. When the coupling is less then 100%, as in the case of a Foucault setup somewhere between the poles and the equator there is an *actual precession* of the pendulum swing.

Overview: the influence of the centripetal force.

The reason that the centripetal force is crucial is the fact that the direction of the centripetal force is not constant.

For contrast: compare with the case of a displacing force that is constant in direction. Take for instance the following setup: a pendulum suspended in a train carriage that accelerates uniformly in a straight line.
That*uniform* acceleration can be incorporated as a tilt of the vertical (just like a plumb line in a uniformly accelerating train carriage will be tilted accordingly); it will not affect the direction of the pendulum swing.

Cumulative

In the case of a Foucault pendulum the centripetal force *does* affect the direction of the plane of swing. While the centripetal force's change of direction during each separate swing is minute, it is nonetheless the determining factor because the effect is *cumulative*.