Fluids in Rigid Body Motion

Introduction

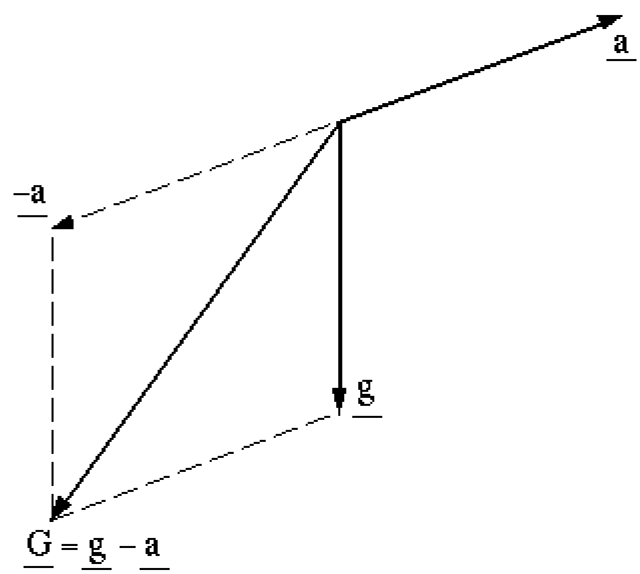
- Recall, for the case of rigid body motion, the equation of motion for fluid flow (the Navier-Stokes equation) reduces to

$$\vec{\nabla} p = \rho(\vec{g} - \vec{a}) \quad \text{or}$$

$$\vec{\nabla} p = \rho \vec{G} \quad \text{where}$$

$$\vec{G} \equiv \vec{g} - \vec{a}$$

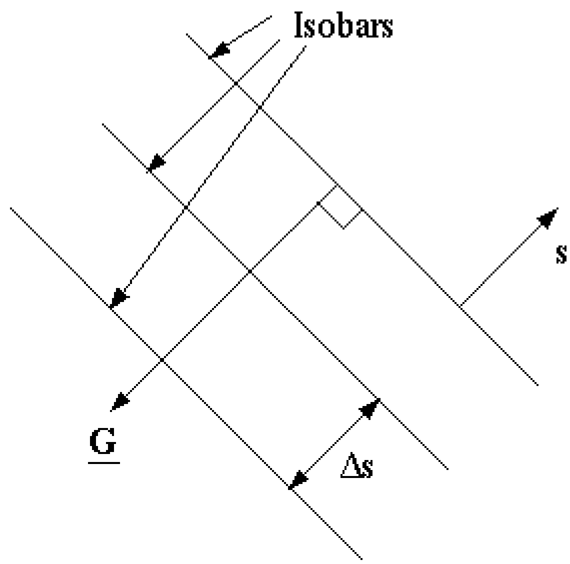
  Note that a new "effective gravity" vector, **G**, has been defined as the vector sum of gravity and the negative of the acceleration vector. This new effective gravity vector can be obtained with a little trigonometry as the resultant vector of adding **g** and **-a**.



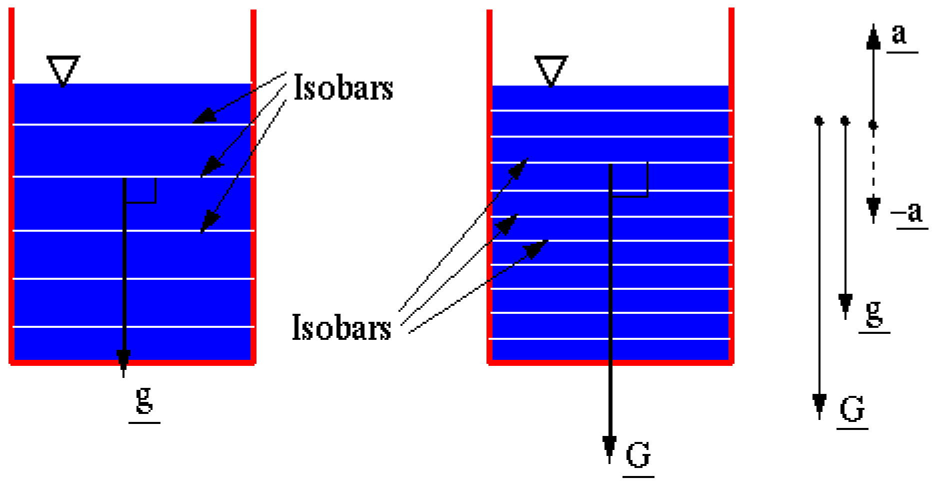
- There are two cases of rigid body motion to be discussed: ***uniform linear rigid body acceleration***, and ***rigid body rotation***.

Uniform Linear Rigid Body Acceleration

- Consider the case where the fluid is accelerated uniformly in some direction. In other words, each fluid particle in the container feels exactly the same acceleration vector, which is constant in time. In such a case, since both the gravity vector and the acceleration vector are constant, the effective gravity vector,**G**, must also be constant.
- Notice, then, that the equation of motion is identical to the hydrostatics equation, except that gravity **g** is replaced by effective gravity **G**. This makes problems of this type no more difficult than simple hydrostatics. In fact, for uniform linear rigid body acceleration, the solution is identical to that of hydrostatics, but with**g** replaced by **G**, and with z (parallel to **g**, i.e. down) replaced by s (parallel to **G**). A good way to remember this is to imagine that the accelerating fluid is instead sitting on a planet where gravity acts in some strange direction (that of vector **G**) and with some magnitude G. One consequence of this is that isobars must be perpendicular to **G**, i.e. perpendicular to direction s. In fact, the pressure increases linearly with distance s, rather than with z.



- As an example, consider a glass of water in an elevator which is accelerating up:

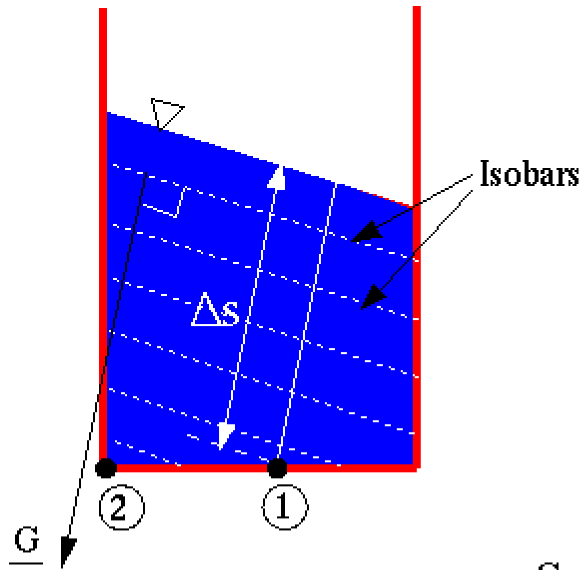


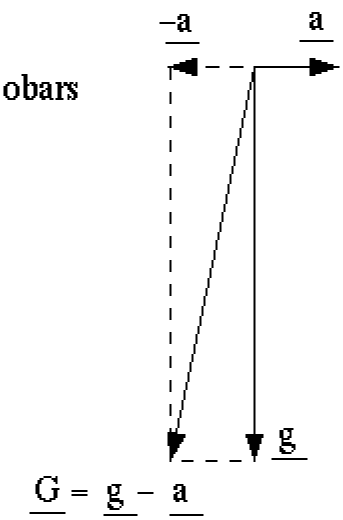
Here, the effective gravity is still downward, but of greater magnitude than g. The isobars are still horizontal surfaces as in hydrostatics. In fact, everything is identical to hydrostatics except for a larger gravity (pretend for example that the glass is sitting on the planet Jupiter). Our simple hydrostatic pressure relationship still applies, but with g replaced by G, and z replaced by s, i.e.

$$p_{below} = p_{above} + \rho G|\Delta s|$$

Also note that "below" and "above" are relative to coordinate s rather than z as well. All else being equal, the pressure at the bottom of the accelerating glass will be greater than that at the bottom of the stationary glass (because G is greater than g).

- Now consider a glass of water in an elevator that accelerates uniformly to the right. Again, the effective gravity vector can be constructed as shown:



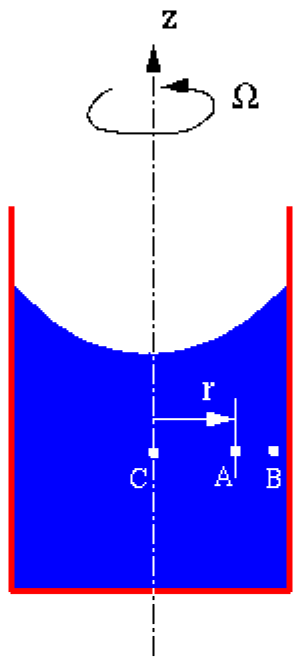


Now, since the effective gravity is tilted at some angle to the lower left, the isobars must be perpendicular to this direction. The isobars are thus titled down and to the right as sketched. Note that the surface remains an isobar of constant pressure p_a, and is therefore also tilted as sketched. At some point 1 in the fluid, the pressure can be found from the revised hydrostatic pressure relationship as follows:

$$p_{below} = p_{above} + \rho G|\Delta s|$$
$$p_1 = p_a + \rho G\Delta s$$

Rigid Body Rotation

- Consider a container of some liquid which is rotating about a vertical axis at some constant angular velocity, as shown in the sketch:



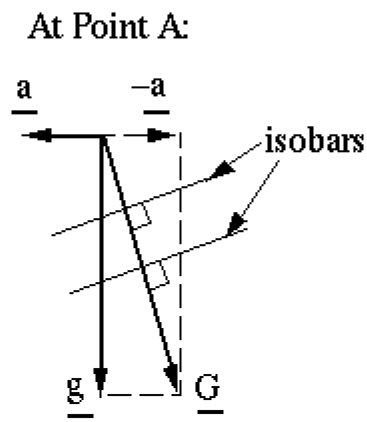
- For any kind of rigid body motion, the equation of motion for fluid flow (the Navier-Stokes equation) reduces to

$$\vec{\nabla}p = \rho(\vec{g} - \vec{a}) \quad \text{or}$$
$$\vec{\nabla}p = \rho\vec{G} \quad \text{where}$$
$$\vec{G} \equiv \vec{g} - \vec{a}$$

In rigid body linear acceleration, effective gravity vector, **G**, was constant everywhere in the fluid. Here, this is no longer the case, since the acceleration of a fluid particle rotating about some axis varies with distance from the axis. In fact, for circular motion, the acceleration is always inward, towards the center of rotation (centripetal acceleration). This acceleration increases linearly with radius (see text for derivation):

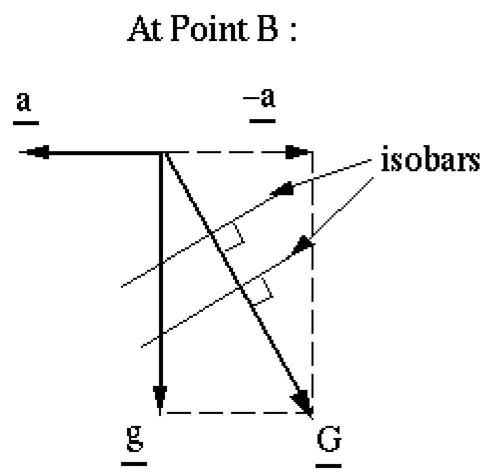
$$a = r \cdot \Omega^2$$

Thus, the effective gravity vector, **G**, is not constant, but varies with radius. In the sketch below, the effective gravity vector at point A is constructed:

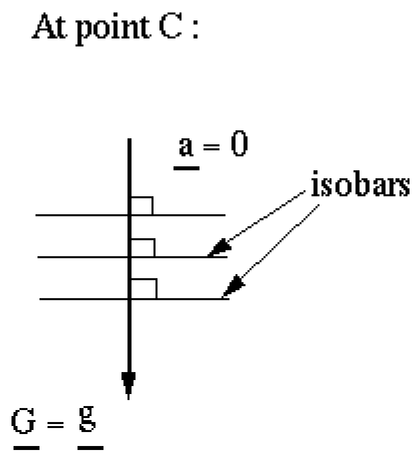


Locally, the isobars near point A are of course still perpendicular to **G**, and they are shown.

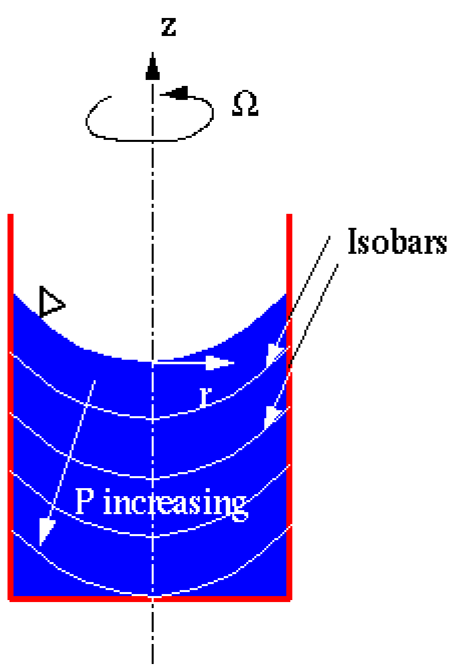
- At other points in the flow, however, the local effective gravity vector is different, since the local acceleration is different. For example, at point B (at a bigger radius than point A) the inward acceleration vector is larger, and **G** tilts further to the right as sketched below:



The local isobars around point B, being perpendicular to the local effective gravity vector, are thus tilted to the upper right even more severely than at point A. At the centerline (point C), the local acceleration is zero, and the effective gravity is identical to the standard gravity; isobars near the centerline of the rotating liquid are horizontal, just as in hydrostatics:



- If this kind of analysis is done everywhere in the flow, the isobars turn out to be paraboloids, which are constructed by rotating a parabola about its axis to generate an axisymmetric surface:



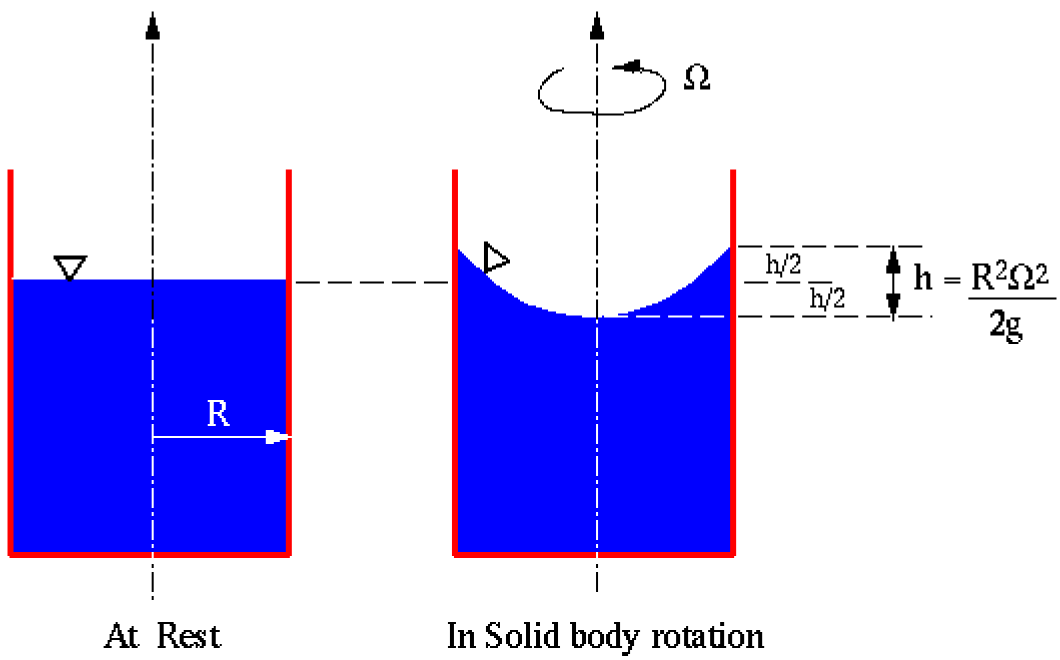
The free surface is of course an isobar, since its pressure is atmospheric. Pressure increases perpendicular to the isobars.

- The text provides a more detailed mathematical derivation of the equations for pressure and for the isobars. If the origin is defined at the lowest point on the free surface, as shown in the sketch above, the equations for pressure and for the isobars are:

$$p = p_a - \rho g z + \tfrac{1}{2}\rho r^2 \Omega^2$$

$$z = \frac{-\left(p - p_a\right)}{\rho g} + \frac{r^2\Omega^2}{2g}$$

- When $p = p_a$, the second equation above becomes an equation describing the free surface. When p is greater than p_a, this equation describes isobars below the surface (at higher pressure). Now denote h as the difference between the height at the center of the free surface and the rim of the free surface. It turns out that, compared to the original free surface (container not rotating at all), the surface dips down at the center an amount (h/2) equal to the amount the surface rises at the rim (also h/2). This is illustrated below:



- Consider a numerical example. A container of water of radius 4.0 inches rotates at a fixed angular velocity of 100 rpm. Estimate h:

$$h = \frac{r^2\Omega^2}{2g} = \frac{\left(\frac{4}{12}\,\mathrm{ft}\right)^2\left[100\,\frac{\mathrm{rot}}{\mathrm{min}}\left(\frac{1\ \mathrm{min}}{60\ \mathrm{sec}}\right)\left(\frac{2\pi\ \mathrm{radians}}{\mathrm{rot}}\right)\right]^2}{2\left(32.2\,\frac{\mathrm{ft}}{\mathrm{s}^2}\right)}$$

$$h = 0.189\ \mathrm{ft.} \quad = \quad 2.3\ \mathrm{in.}$$

A simple experiment with a rotating cup of water can show that this prediction is quite accurate.

- Notice that density does not appear in the equation for the isobars or in the equation for h. Thus, the water could be replaced with any other liquid, and the result would be identical. Pressure would increase more rapidly with depth for a denser liquid, but the shape of the free surface would stay the same, regardless of the liquid used. Furthermore, since this is rigid body rotation, portions of the liquid could even be removed or replaced with solid material. In this way, any rotating chunk of liquid, regardless of its shape, can be analyzed as if it were part of a big container rotating about the z-axis as in the figures above.

Source:

http://www.mne.psu.edu/cimbala/Learning/Fluid/Rigid_body/rigid_body.htm