Equilibrium of Stress

Consider two perpendicular planes passing through a point *p*. The stress components acting on these planes are as shown in Fig. 3.4.1a. These stresses are usually shown together acting on a small material element of finite size, Fig. 3.4.1b. It has been seen that the stress may vary from point to point in a material but, if the element is very small, the stresses on one side can be taken to be (more or less) equal to the stresses acting on the other side. By convention, in analyses of the type which will follow, all stress components shown are *positive*.

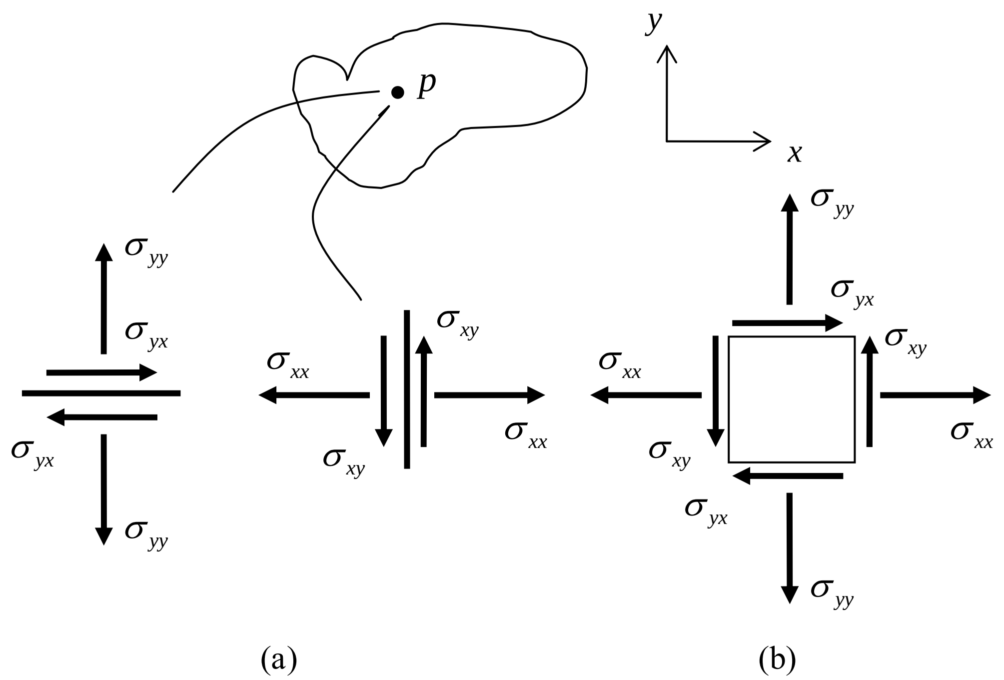

Figure 3.4.1: stress components acting on two perpendicular planes through a point; (a) two perpendicular surfaces at a point, (b) small material element at the point

The four stresses can conveniently be written in the matrix form:

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} \quad (3.4.1)$$

It will be shown below that the stress components acting on *any* other plane through *p* can be evaluated from a knowledge of only these stress components.

3.4.1 Symmetry of the Shear Stress

Consider the material element shown in Fig. 3.4.1b, reproduced in Fig. 3.4.2a below. The element has dimensions is $\Delta x \times \Delta y$ and is subjected to uniform stresses over its sides. The resultant forces of the stresses acting on each side of the element act through the side-centres, and are shown in Fig. 3.4.2b. The stresses shown are positive, but note how *positive stresses can lead to negative forces*, depending on the definition of the $x-y$ axes used. The resultant force on the complete element is seen to be zero.

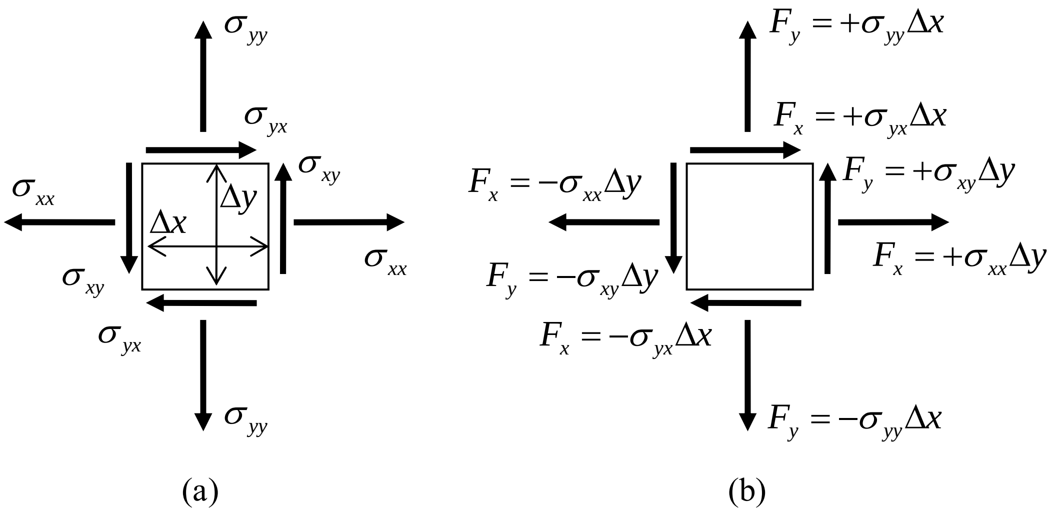

Figure 3.4.2: stress components acting on a material element; (a) stresses, (b) resultant forces on each side

By taking moments about any point in the block, one finds that {▲Problem 1}

$$\sigma_{xy} = \sigma_{yx} \tag{3.4.2}$$

Thus the shear stresses acting on the element are all equal, and for this reason the σ_{yx} stresses are usually labelled σ_{xy}, Fig. 3.4.3a, or simply labelled τ, Fig. 3.4.3b.

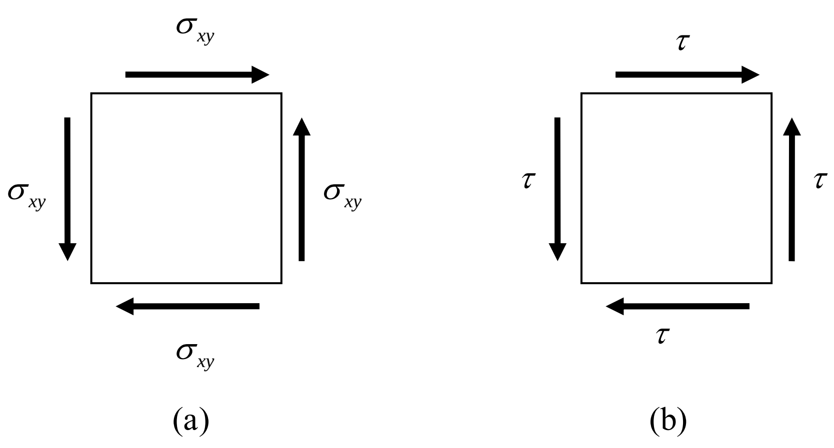

Figure 3.4.3: shear stress acting on a material element

3.4.2 Three Dimensional Stress

The three-dimensional counterpart to the two-dimensional element of Fig. 3.4.2 is shown in Fig. 3.4.4. Again, all stresses shown are positive.

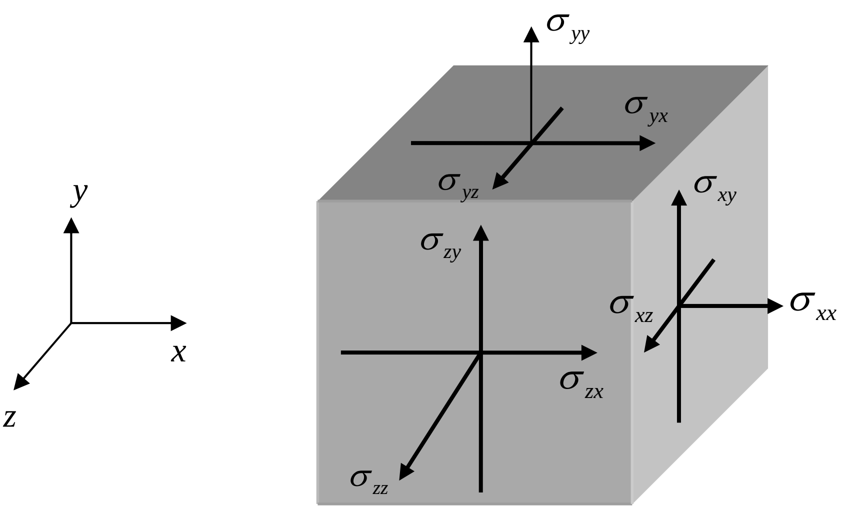

Figure 3.4.4: a three dimensional material element

Moment equilibrium in this case requires that

$$\sigma_{xy} = \sigma_{yx}, \quad \sigma_{xz} = \sigma_{zx}, \quad \sigma_{yz} = \sigma_{zy} \tag{3.4.3}$$

The nine stress components, six of which are independent, can now be written in the matrix form

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \tag{3.4.4}$$

A vector **F** has one direction associated with it and is characterised by *three* components (F_x, F_y, F_z). The stress is a quantity which has two directions associated with it (the direction of a force and the normal to the plane on which the force acts) and is characterised by the *nine* components of Eqn. 3.4.4. Such a mathematical object is called a **tensor**. Just as the three components of a vector change with a change of coordinate axes (for example, as in Fig. 2.2.1), so the nine components of the **stress tensor** change with a change of axes. This is discussed in the next section for the two-dimensional case. (The concept of a tensor will be examined more closely in Books II and especially IV.)

3.4.3 Stress Transformation Equations

Consider the case where the nine stress components acting on three perpendicular planes through a material particle are known. These components are σ_{xx}, σ_{xy}, etc. when using x, y, z axes, and can be represented by the cube shown in Fig. 3.4.5a. Rotate now the planes about the three axes – these new planes can be represented by the rotated cube shown in Fig. 3.4.5b; the axes normal to the planes are now labelled x', y', z' and the corresponding stress components with respect to these new axes are $\sigma'_{xx}, \sigma'_{xy}$, etc.

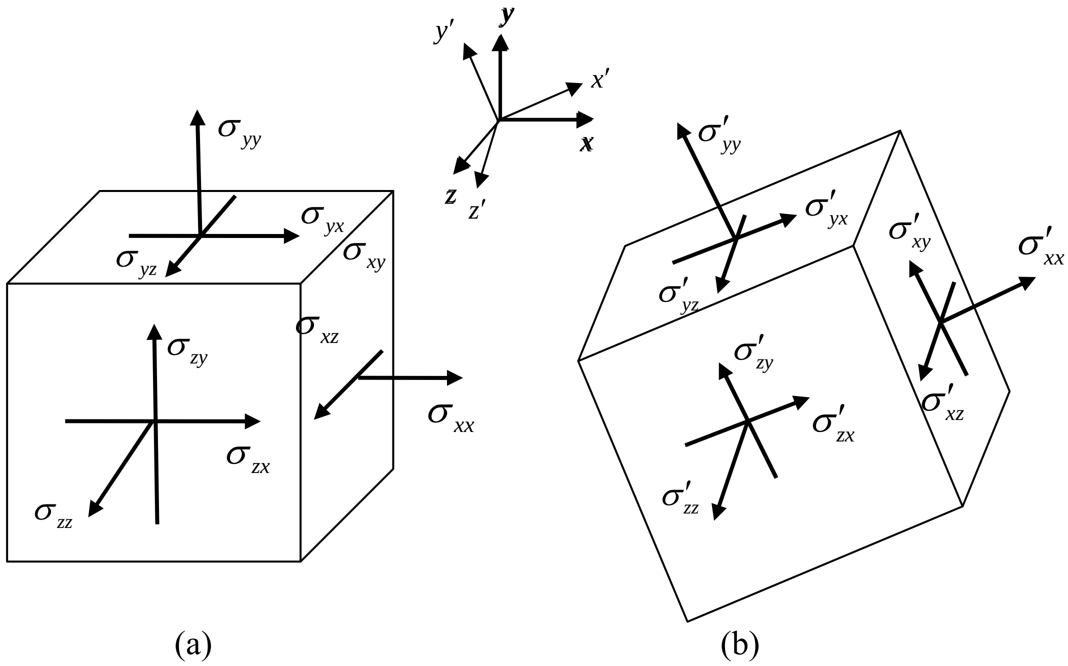

Figure 3.4.5: a three dimensional material element; (a) original element, (b) rotated element

There is a relationship between the stress components σ_{xx}, σ_{xy}, etc. and the stress components $\sigma'_{xx}, \sigma'_{xy}$, etc. The relationship can be derived using Newton's Laws. The equations describing the relationship in the fully three-dimensional case are very lengthy – they will be discussed in Books II and IV. Here, the relationship for the two-dimensional case will be derived – this 2D relationship will prove very useful in analysing many practical situations.

Two-dimensional Stress Transformation Equations

Assume that the stress components of Fig. 3.4.6a are known. It is required to find the stresses arising on other planes through *p*. Consider the perpendicular planes shown in Fig. 3.4.65b, obtained by rotating the original element through a positive (counterclockwise) angle θ. The new surfaces are defined by the axes $x' - y'$.

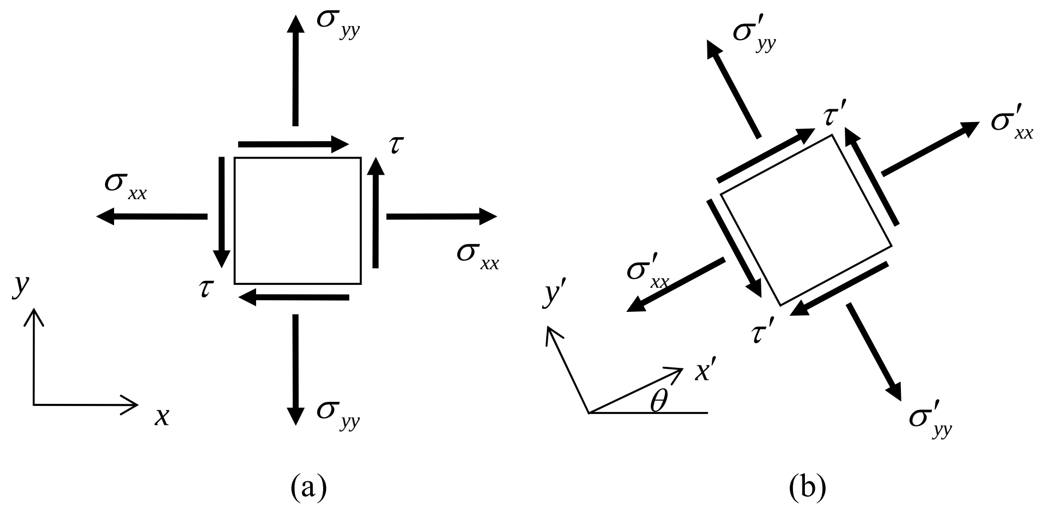

Figure 3.4.6: stress components acting on two different sets of perpendicular surfaces, i.e. in two different coordinate systems; (a) original system, (b) rotated system

To evaluate these new stress components, consider a triangular element of material at the point, Fig. 3.4.7. Carrying out force equilibrium in the direction x', one has (with unit depth into the page)

$$\sum F_{x'}: \quad \sigma'_{xx}|AB| - \sigma_{xx}|OB|\cos\theta - \sigma_{yy}|OA|\sin\theta - \tau|OB|\sin\theta - \tau|OA|\cos\theta = 0 \qquad (3.4.5)$$

Since $|OB| = |AB|\cos\theta,\ |OA| = |AB|\sin\theta$, and dividing through by $|AB|$,

$$\sigma'_{xx} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + \tau\sin 2\theta \qquad (3.4.6)$$

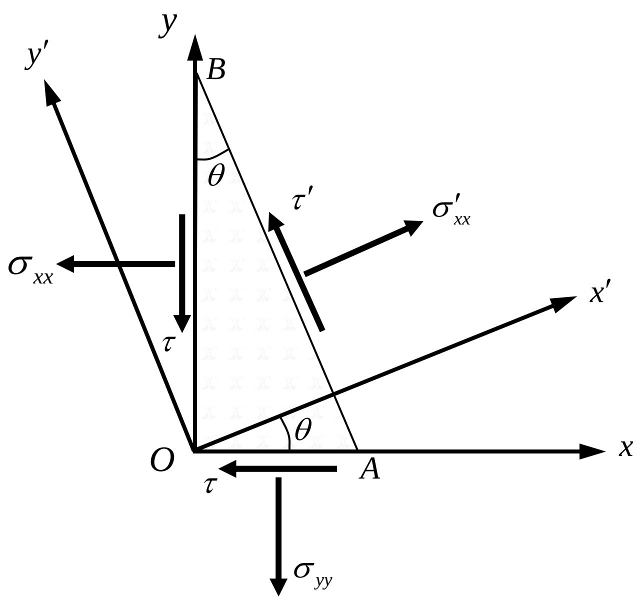

Figure 3.4.7: a free body diagram of a triangular element of material

The forces can also be resolved in the y' direction and one obtains the relation

$$\tau' = (\sigma_{yy} - \sigma_{xx})\sin\theta\cos\theta + \tau\cos 2\theta \tag{3.4.7}$$

Finally, consideration of the element in Fig. 3.4.8 yields two further relations, one of which is the same as Eqn. 3.4.6.

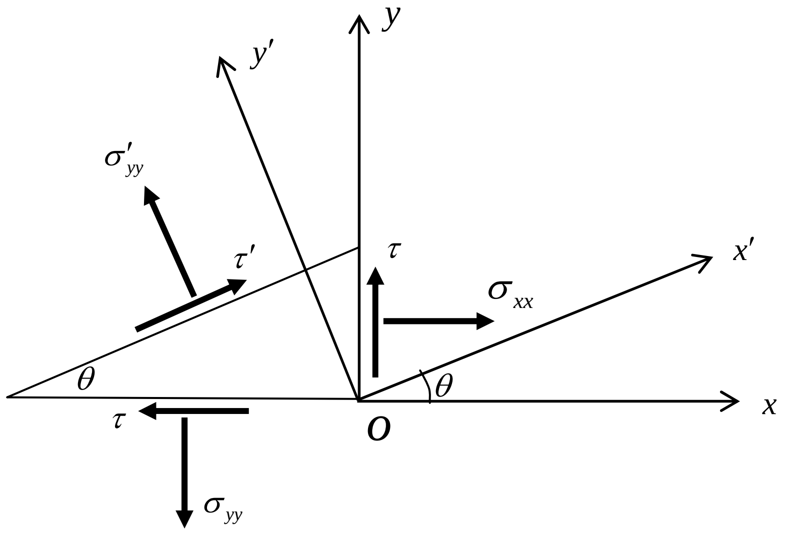

Figure 3.4.8: a free body diagram of a triangular element of material

In summary, one obtains the **stress transformation equations**:

$$\begin{aligned}
\sigma'_{xx} &= \cos^2\theta\sigma_{xx} + \sin^2\theta\sigma_{yy} + \sin 2\theta\sigma_{xy} \\
\sigma'_{yy} &= \sin^2\theta\sigma_{xx} + \cos^2\theta\sigma_{yy} - \sin 2\theta\sigma_{xy} \\
\sigma'_{xy} &= \sin\theta\cos\theta(\sigma_{yy} - \sigma_{xx}) + \cos 2\theta\sigma_{xy}
\end{aligned}$$

2D Stress Transformation Equations (3.4.8)

These equations have many uses, as will be seen in the next section.

In matrix form,

$$\begin{bmatrix} \sigma'_{xx} & \sigma'_{xy} \\ \sigma'_{yx} & \sigma'_{yy} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{3.4.9}$$

Body Force, Acceleration and Non-Uniform Stress

Here, it will be shown that the Stress Transformation Equations are valid also when (i) there are body forces, (ii) the body is accelerating and (iii) the stress and other quantities are not uniform.

Suppose that a body force $\mathbf{F}_b = (\mathbf{F}_b)_x \mathbf{i} + (\mathbf{F}_b)_y \mathbf{j}$ acts on the material and that the material is accelerating with an acceleration $\mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j}$. The components of body force and acceleration are shown in Fig. 3.4.9 (a reproduction of Fig. 3.4.7). The body force will vary depending on the size of the material under consideration, e.g. the force of gravity $\mathbf{F}_b = m\mathbf{g}$ will be larger for larger materials; therefore consider a quantity which is independent of the amount of material: the body force per unit mass, $\mathbf{F}_b / m$. Then, Eqn 3.4.5 now reads

$$\begin{aligned}\sum F_{x'}: \quad & \sigma'_{xx}|AB| - \sigma_{xx}|OB|\cos\theta - \sigma_{yy}|OA|\sin\theta - \tau|OB|\sin\theta - \tau|OA|\cos\theta \\ & + (\mathbf{F}_b / m)_x m\cos\theta + (\mathbf{F}_b / m) m\sin\theta + m a_x \cos\theta + m a_y \sin\theta = 0\end{aligned} \tag{3.4.10}$$

where m is the mass of the triangular portion of material. The volume of the triangle is $|AB|^2 / \sin 2\theta$ so that, this time, when 3.4.10 is divided through by $|AB|$, one has

$$\begin{aligned}\sigma'_{xx} &= \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + \tau\sin 2\theta \\ &\quad - |AB|\rho\left\{(\mathbf{F}_b / m)_x / 2\sin\theta + (\mathbf{F}_b / m) / 2\cos\theta + a_x / 2\sin\theta + a_y / 2\cos\theta\right\}\end{aligned} \tag{3.4.11}$$

where ρ is the density. Now, as the element is shrunk in size down to the vertex O, $|AB| \to 0$, and Eqn. 3.4.6 is recovered. Thus the Stress Transformation Equations are valid provided the material under consideration is very small; in the limit, they are valid "at the point" O.

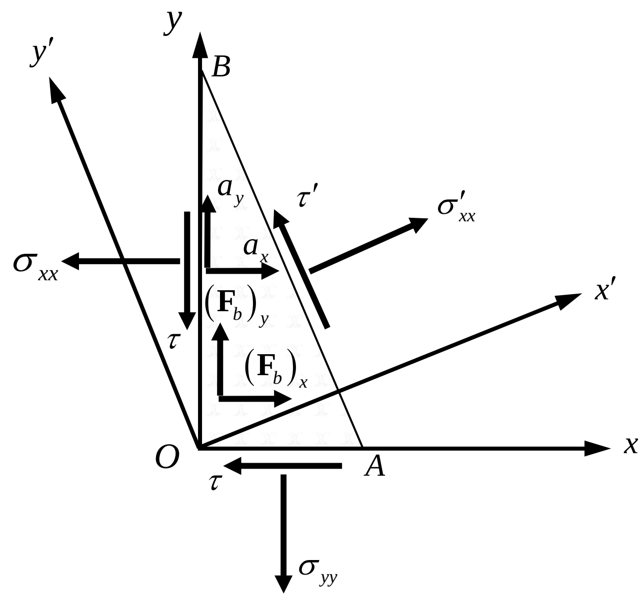

Figure 3.4.9: a free body diagram of a triangular element of material, including a body force and acceleration

Finally, consider the case where the stress is not uniform over the faces of the triangular portion of material. Intuitively, it can be seen that, if one again shrinks the portion of material down in size to the vertex O, the Stress Transformation Equations will again be valid, with the quantities $\sigma'_{xx}, \sigma_{xx}, \sigma_{yy}$ etc. being the values "at" the vertex. To be more precise, consider the σ_{xx} stress acting over the face $|OB|$ in Fig. 3.4.10. No matter how the stress varies in the material, if the distance $|OB|$ is small, the stress can be approximated by a linear stress distribution, Fig. 3.4.10b. This linear distribution can itself be decomposed into two components, a uniform stress of magnitude σ^o_{xx} (the value of σ_{xx} at the vertex) and a triangular distribution with maximum value $\Delta\sigma_{xx}$. The resultant force on the face is then $|OB|\left(\sigma^o_{xx} + \Delta\sigma_{xx}/2\right)$. This time, as the element is shrunk in size, $\Delta\sigma_{xx} \to 0$ and Eqn. 3.4.6 is again recovered. The same argument can be used to show that the Stress Transformation Equations are valid for any varying stress, body force or acceleration.

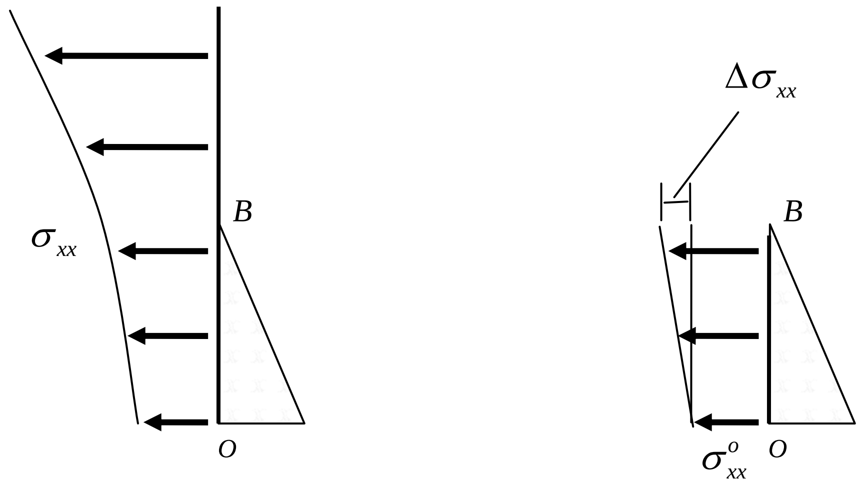

Figure 3.4.10: stress varying over a face; (a) stress is linear over OB if OB is small, (b) linear distribution of stress as a uniform stress and a triangular stress

Three Dimensions Re-visited

As the planes were rotated in the two-dimensional analysis, no consideration was given to the stresses acting in the "third dimension". Considering again a three dimensional block, Fig. 3.4.11, there is only one traction vector acting *on* the $x-y$ plane at the material particle, **t**. This traction vector can be described in terms of the x, y, z axes as $\mathbf{t} = \sigma_{zx}\mathbf{i} + \sigma_{zy}\mathbf{j} + \sigma_{zz}\mathbf{k}$, Fig 3.4.11a. Alternatively, it can be described in terms of the x', y', z' axes as $\mathbf{t} = \sigma'_{zx}\mathbf{i}' + \sigma'_{zy}\mathbf{j}' + \sigma'_{zz}\mathbf{k}'$, Fig 3.4.11b.

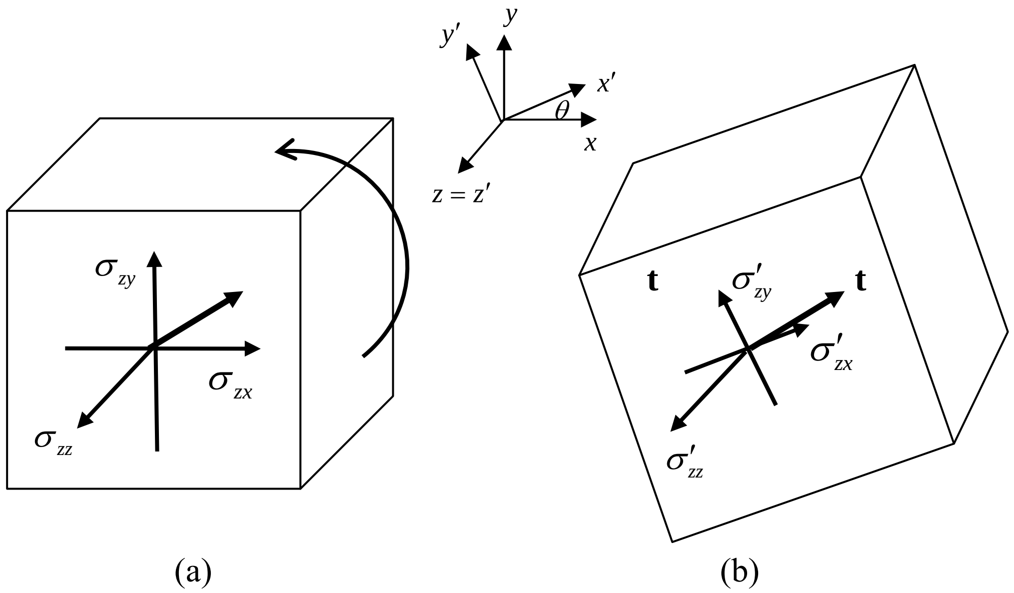

Figure 3.4.11: a three dimensional material element; (a) original element, (b) rotated element (rotation about the *z* axis)

With the rotation only happening *in* the $x-y$ plane, about the *z* axis, one has $\sigma_{zz}=\sigma'_{zz}$, $\mathbf{k}=\mathbf{k}'$. One can thus examine the two dimensional $x-y$ plane shown in Fig. 3.4.12, with

$$\sigma_{zx}\mathbf{i}+\sigma_{zy}\mathbf{j}=\sigma'_{zx}\mathbf{i}'+\sigma'_{zy}\mathbf{j}'. \tag{3.4.12}$$

Using some trigonometry, one can see that

$$\begin{aligned}\sigma'_{zx}&=+\sigma_{zx}\cos\theta+\sigma_{zy}\sin\theta\\ \sigma'_{zy}&=-\sigma_{zx}\sin\theta+\sigma_{zy}\cos\theta\end{aligned}. \tag{3.4.13}$$

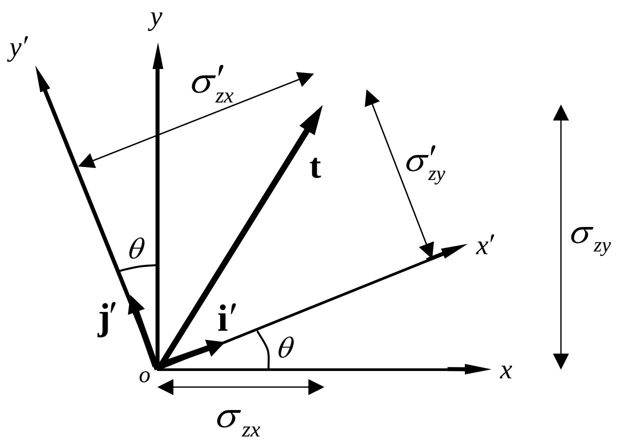

Figure 3.4.12: the traction vector represented using two different coordinate systems

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