Equations of Motion

In Part I, balance of forces and moments acting on any component was enforced in order to ensure that the component was in equilibrium. Here, allowance is made for stresses which vary continuously throughout a material, and force equilibrium of any portion of material is enforced.

One-Dimensional Equation

Consider a one-dimensional differential element of length Δx and cross sectional area A, Fig. 1.1.1. Let the *average* body force per unit volume acting on the element be b and the *average* acceleration and density of the element be a and ρ. Stresses σ act on the element.

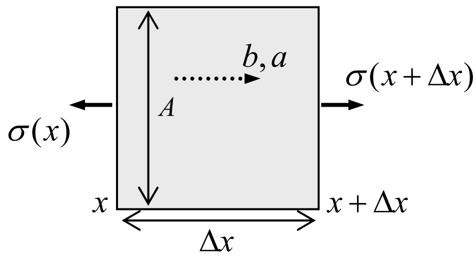

Figure 1.1.1: a differential element under the action of surface and body forces

The net surface force acting is $\sigma(x+\Delta x)A - \sigma(x)A$. If the element is small, then the body force and velocity can be assumed to vary linearly over the element and the average will act at the centre of the element. Then the body force acting on the element is $Ab\Delta x$ and the inertial force is $\rho A \Delta x a$. Applying Newton's second law leads to

$$\begin{aligned} &\sigma(x+\Delta x)A - \sigma(x)A + b\Delta x A = \rho a \Delta x A \\ &\rightarrow \frac{\sigma(x+\Delta x) - \sigma(x)}{\Delta x} + b = \rho a \end{aligned} \tag{1.1.1}$$

so that, by the definition of the derivative, in the limit as $\Delta x \rightarrow 0$,

$$\boxed{\frac{d\sigma}{dx} + b = \rho a} \quad \textbf{1-d Equation of Motion} \tag{1.1.2}$$

which is the one-dimensional **equation of motion**. Note that this equation was derived on the basis of a physical law and must therefore be satisfied for all materials, whatever they be composed of.

The derivative $d\sigma / dx$ is the **stress gradient** – physically, it is a measure of how rapidly the stresses are changing.

Example

Consider a bar of length l which hangs from a ceiling, as shown in Fig. 1.1.2.

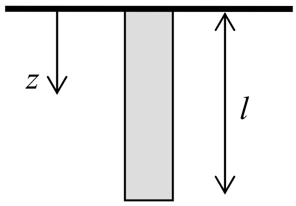

Figure 1.1.2: a hanging bar

The gravitational force is $F = mg$ downward and the body force per unit volume is thus $b = \rho g$. There are no accelerating material particles. Taking the *z* axis positive down, an integration of the equation of motion gives

$$\frac{d\sigma}{dz} + \rho g = 0 \quad \rightarrow \quad \sigma = -\rho g z + c \tag{1.1.3}$$

where *c* is an arbitrary constant. The lower end of the bar is free and so the stress there is zero, and so

$$\sigma = \rho g (l - z) \tag{1.1.4}$$

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Two-Dimensional Equations

Consider now a two dimensional infinitesimal element of width and height Δx and Δy and unit depth (into the page).

Looking at the normal stress components acting in the $x-$direction, and allowing for variations in stress over the element surfaces, the stresses are as shown in Fig. 1.1.3.

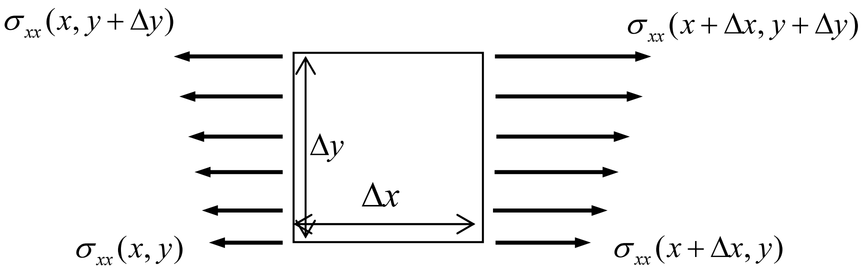

Figure 1.1.3: varying stresses acting on a differential element

Using a (two dimensional) Taylor series and dropping higher order terms then leads to the linearly varying stresses illustrated in Fig. 1.1.4. (where $\sigma_{xx} \equiv \sigma_{xx}(x, y)$ and the partial derivatives are evaluated at (x, y)), which is a reasonable approximation when the element is small.

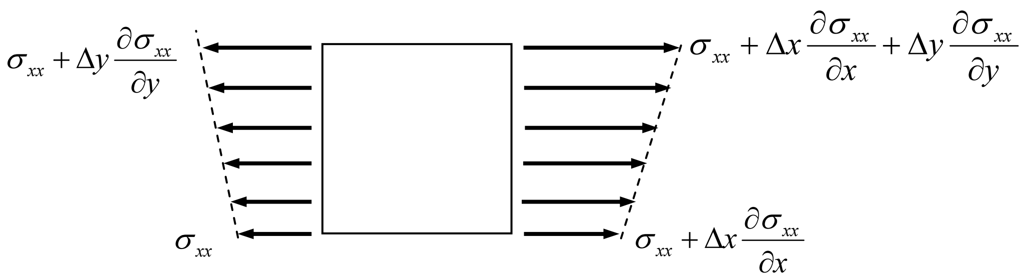

Figure 1.1.4: linearly varying stresses acting on a differential element

The effect (resultant force) of this linear variation of stress on the plane can be replicated by a *constant* stress acting over the whole plane, the size of which is the *average* stress. For the left and right sides, one has, respectively,

$$\sigma_{xx} + \frac{1}{2}\Delta y \frac{\partial \sigma_{xx}}{\partial y}, \quad \sigma_{xx} + \Delta x \frac{\partial \sigma_{xx}}{\partial x} + \frac{1}{2}\Delta y \frac{\partial \sigma_{xx}}{\partial y} \tag{1.1.5}$$

One can take away the stress $(1/2)\Delta y \partial \sigma_{xx} / \partial y$ from both sides without affecting the net force acting on the element so one finally has the representation shown in Fig. 1.1.5.

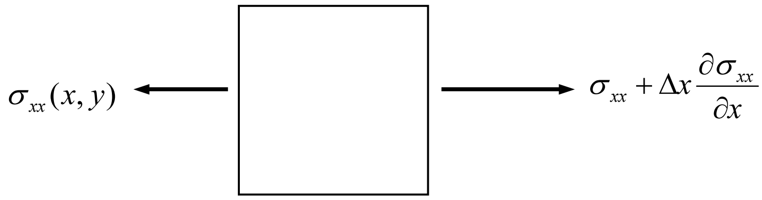

Figure 1.1.5: net stresses acting on a differential element

Carrying out the same procedure for the shear stresses contributing to a force in the x −direction leads to the stresses shown in Fig. 1.1.6.

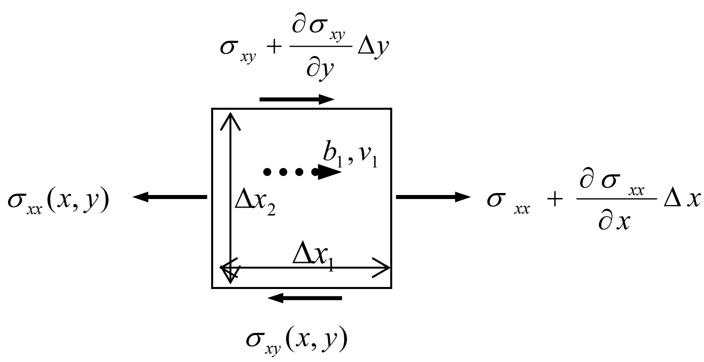

Figure 1.1.6: normal and shear stresses acting on a differential element

Take a_x, b_x to be the average acceleration and body force, and ρ to be the average density. Newton's law then yields

$$-\sigma_{xx}\Delta y+\left(\sigma_{xx}+\Delta x\frac{\partial\sigma_{xx}}{\partial x}\right)\Delta y-\sigma_{xy}\Delta x+\left(\sigma_{xy}+\Delta y\frac{\partial\sigma_{xy}}{\partial y}\right)\Delta y+b_x\Delta x\Delta y=\rho a_x\Delta x\Delta y$$

(1.1.6)

which, dividing through by $\Delta x\Delta y$ and taking the limit, gives

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+b_x=\rho a_x \tag{1.1.7}$$

A similar analysis for force components in the $y-$direction yields another equation and one then has the two-dimensional equations of motion:

$$\boxed{\begin{aligned}\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+b_x&=\rho a_x\\ \frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+b_y&=\rho a_y\end{aligned}}$$

2-D Equations of Motion (1.1.8)

Three-Dimensional Equations

Similarly, one can consider a three-dimensional element, and one finds that

$$\boxed{\begin{aligned}\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+\frac{\partial\sigma_{xz}}{\partial z}+b_x&=\rho a_x\\ \frac{\partial\sigma_{yx}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{yz}}{\partial z}+b_y&=\rho a_y\\ \frac{\partial\sigma_{zx}}{\partial x}+\frac{\partial\sigma_{zy}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}+b_z&=\rho a_z\end{aligned}}$$

3-D Equations of Motion (1.1.9)

These three equations express force-balance in, respectively, the x, y, z directions.

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signant par $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ les projections algébriques de la force accélératrice qui serait capable de produire à elle seule le mouvement effectif d'une particule, et prenant x, y, z, t pour variables indépendantes, on obtiendra, à la place des équations (1), celles qui suivent

$$(2)\quad \left\{ \begin{array}{l} \frac{dA}{dx}+\frac{dF}{dy}+\frac{dE}{dz}+\rho X=\rho\mathfrak{X}, \\ \frac{dF}{dx}+\frac{dB}{dy}+\frac{dD}{dz}+\rho Y=\rho\mathfrak{Y}, \\ \frac{dE}{dx}+\frac{dD}{dy}+\frac{dC}{dz}+\rho Z=\rho\mathfrak{Z}. \end{array} \right.$$

Enfin, si l'on nomme ξ, η, ζ les déplacements de la particule qui, au bout d'un temps t, coïncide avec le point (x, y, z), mesurés parallèlement aux axes coordonnés, on trouvera, en supposant ces déplacements très-petits,

Figure 1.1.7: from Cauchy's Exercices de Mathematiques (1829)

The Equations of Equlibrium

If the material is not moving (or is moving at constant velocity) and is in static equilibrium, then the equations of motion reduce to the **equations of equilibrium**,

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{xy}}{\partial y}+\frac{\partial \sigma_{xz}}{\partial z}+b_x &= 0 \\ \frac{\partial \sigma_{yx}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}+\frac{\partial \sigma_{yz}}{\partial z}+b_y &= 0 \\ \frac{\partial \sigma_{zx}}{\partial x}+\frac{\partial \sigma_{zy}}{\partial y}+\frac{\partial \sigma_{zz}}{\partial z}+b_z &= 0 \end{aligned}$$

3-D Equations of Equilibrium (1.1.10)

These equations express the force balance between surface forces and body forces in a material. The equations of equilibrium may also be used as a good approximation in the analysis of materials which have relatively small accelerations.