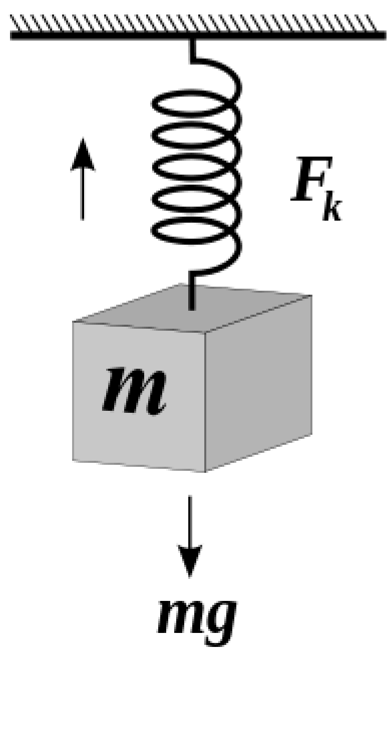
Dynamical Systems

What is a dynamical system?

Systems can be characterized by the specific relation between their input(s) and output(s). A static system has an output that only depends on its input. A mechanical example of such a system is an idealized, massless (mass=0) spring. The length of the spring depends only on the force (the input) that acts upon it. Change the input force, and the length of the spring will change, and this will happen instantaneously (obviously a massless spring is a theoretical construct). A system becomes dynamical (it is said to have *dynamics* when a mass is attached to the spring. Now the position of the mass (and equivalently, the length of the spring) is no longer directly dependent on the input force, but is also tied to the acceleration of the mass, which in turn depends on the sum of all forces acting upon it (the sum of the input force and the force due to the spring). The net force depends on the position of the mass, which depends on the length of the spring, which depends on the spring force. The property that acceleration of the mass depends on its position makes this a dynamical system.



A spring with a mass attached

Dynamical systems can be characterized by differential equations that relate the state derivatives (e.g. velocity or acceleration) to the state variables (e.g. position). The differential equation for the spring-mass system depicted above is:

[Math Processing Error]

Where *[Math Processing Error]* is the position of the mass *[Math Processing Error]* (the length of the spring), *[Math Processing Error]* is the second derivative of position (i.e. acceleration), *[Math Processing Error]* is a constant (related to the stiffness of the spring), and *[Math Processing Error]* is the graviational constant.

The system is said to be a *second order* system, as the highest derivative that appears in the differential equation describing the system, is two. The position *[Math Processing Error]* and its time derivative *[Math Processing Error]* are called *states* of the system, and *[Math Processing Error]* and *[Math Processing Error]* are called *state derivatives*.

Most systems out there in nature are dynamical systems. For example most chemical reactions under natural circumstances are dynamical: the rate of change of a chemical reaction depends on the amount of chemical present, in other words the state derivative is proportional to the state. Dynamical systems exist in biology as well. For example the rate of change of a certain species depends on its population size.

Dynamical equations are often described by a set of *coupled differential equations*. For example, the reproduction rate of rabbits (state derivative 1) depends on the population of rabbits (state 1) and on the population size of foxes (state 2). The reproduction rate of foxes (state derivative 2) depends on the population of foxes (state 2) and also on the population of rabbits (state 1). In this case we have two coupled first-order differential equations, and hence a system of order two. The so-called predator-prey model is also known as the Lotka-Volterra equations.

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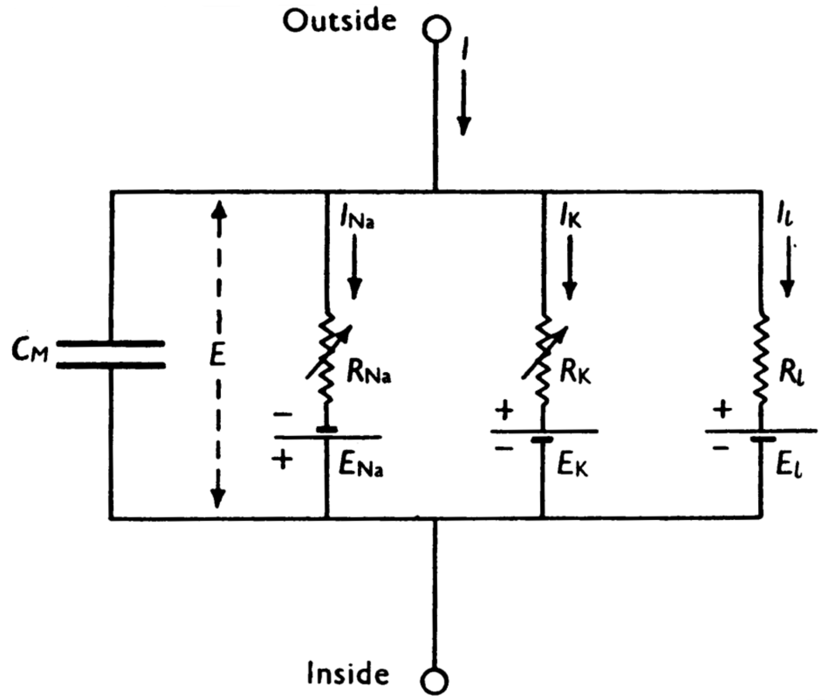
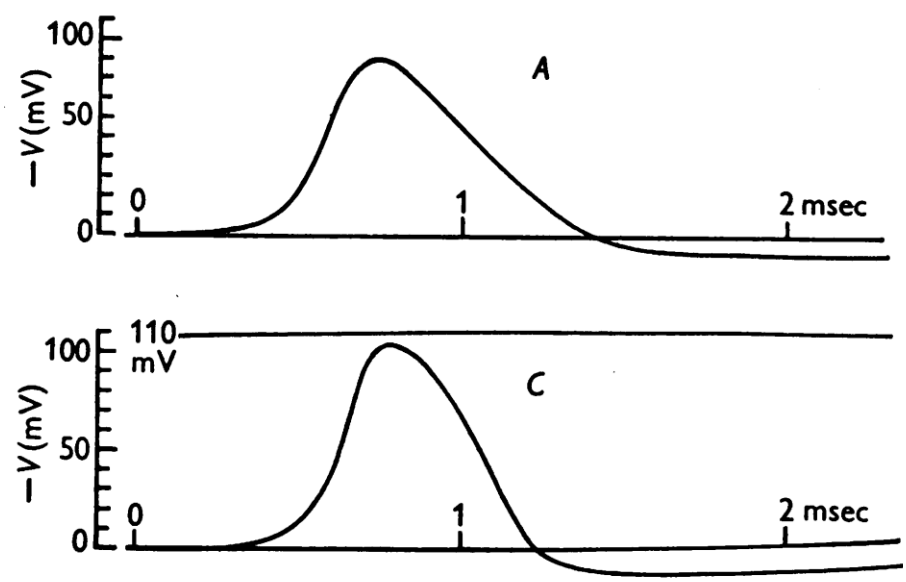
Why make models?

There are two main reasons: one being practical and one mostly theoretical. The practical use is prediction. A typical example of a dynamical system that is modelled for prediction is the weather. The weather is a very complex, (high-order, nonlinear, coupled and chaotic) system. More theoretically, one reason to make models is to test the validity of a functional hypothesis of an observed phenomenon. A beautiful example is the model made by Hodgkin and Huxley to understand how action potentials arise and propagate in neurons [1][2]. They modelled the different (voltage-gated) ion channels in an axon membrane and showed using mathematical models that indeed the changes in ion concentrations were responsible for the electical spikes observed experimentally 7 years earlierr.



Hodgkin-Huxley model of voltage-gated ion channels



Action potentials across the membrane

A second theoretical reason to make models is that it is sometimes very difficult, if not impossible, to answer a certain question empirically. As an example we take the following biomechanical question: Would you be able to jump higher if your biceps femoris (part of your hamstrings) were two separate muscles each crossing only one joint rather than being one muscle crossing both the hip and knee joint? Not a strange question as one could then independently control the torques around each joint.

In order to answer this question empirically, one would like to do the following experiment:

- measure the maximal jump height of a subject
- change only the musculoskeletal properties in question
- measure the jump height again

Of course, such an experiment would yield several major ethical, practical and theoretical drawbacks. It is unlikely that an ethics committee would approve the transplantation of the origin and insertion of the hamstrings in order to examine its effect on jump height. And even so, one would have some difficulties finding subjects. Even with a volunteer for such a surgery it would not bring us any closer to an answer. After such a surgery, the subject would not be able to jump right away, but has to undergo severe revalidation and surely during such a period many factors will undesirably change like maximal contractile forces. And even if the subject would fully recover (apart from the hamstrings transplantation), his or her nervous system would have to find the new optimal muscle stimulation pattern.

If one person jumps lower than another person, is that because she cannot jump as high with her particular muscles, or was it just that her CNS was not able to find the optimal muscle activation pattern? Ultimately, one wants to know through what mechanism the subject's jump performance changes. To investigate this, one would need to know, for example, the forces produced by the hamstrings as a function of time, something that is impossible to obtain experimentally. Of course, this example is somewhat ridiculous, but its message is hopefully clear that for several questions a strict empirical approach is not suitable. An alternative is provided by mathematical modelling.

Next steps

In the next topic, we will be examining three systems — a mass-spring system, a system representing weather patterns, and a system characterizing predator-prey interactions. In each case we will see how to go from differential equations characterizing the dynamics of the system, to Python code, and run that code to simulate the behaviour of the system over time. We will see the great power of simulation, namely the ability to change aspects of the system at will, and simulate to explore the resulting change in system behaviour.

[next]

References

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[2] A. L. Hodgkin, A. F. Huxley, A. L. Hodgkin, and A. F. Huxley. A quantitative description of membrane current and its application to conduction and excitation in nerve. 1952.*Bull. Math. Biol.*, 52(1-2):25-71, 1990. [bib]

[3] O. Ekeberg, P. Wallen, A. Lansner, H. Traven, L. Brodin, and S. Grillner. A computer based model for realistic simulations of neural networks. I. The single neuron and synaptic interaction. *Biol Cybern*, 65(2):81-90, 1991. [bib]

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