Instructional Objectives

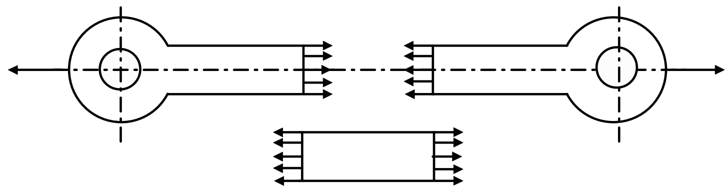
At the end of this lesson, the students should be able to understand

- Types of loading on machine elements and allowable stresses.
- Concept of yielding and fracture.
- Different theories of failure.
- Construction of yield surfaces for failure theories.
- Optimize a design comparing different failure theories

3.1.1Introduction

Machine parts fail when the stresses induced by external forces exceed their strength. The external loads cause internal stresses in the elements and the component size depends on the stresses developed. Stresses developed in a link subjected to uniaxial loading is shown in **figure-3.1.1.1**. Loading may be due to:

a) The energy transmitted by a machine element.
b) Dead weight.
c) Inertial forces.
d) Thermal loading.
e) Frictional forces.



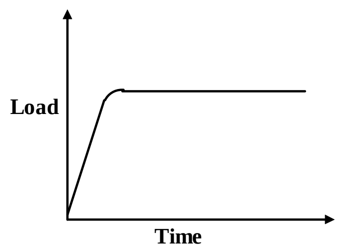
***3.1.1.1A**- Stresses developed in a link subjected to uniaxial loading*

In another way, load may be classified as:

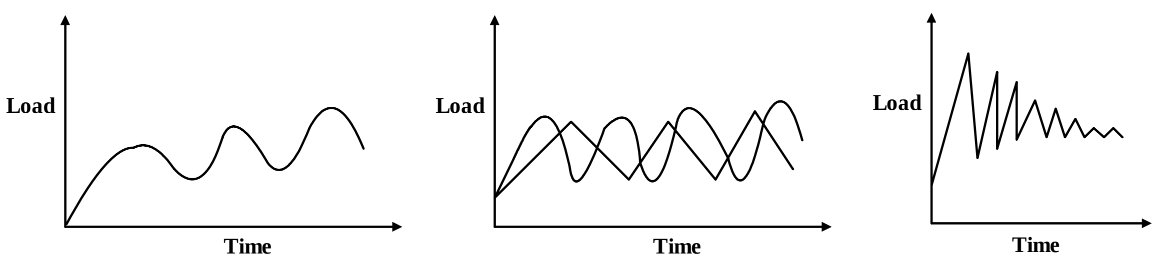
a) Static load- Load does not change in magnitude and direction and normally increases gradually to a steady value.

b) Dynamic load- Load may change in magnitude for example, traffic of varying weight passing a bridge.Load may change in direction, for example, load on piston rod of a double acting cylinder.

Vibration and shock are types of dynamic loading. **Figure-3.1.1.2** shows load vs time characteristics for both static and dynamic loading of machine elements.



Static Loading



Dynamic Loading

***3.1.1.2F** - Types of loading on machine elements.*

3.1.2 Allowable Stresses: Factor of Safety

Determination of stresses in structural or machine components would be meaningless unless they are compared with the material strength. If the induced stress is less than or equal to the limiting material strength then the designed component may be considered to be safe and an indication about the size of the component is obtained. The strength of various materials for engineering applications is determined in the laboratory with standard specimens. For example, for tension and compression tests a round rod of specified dimension is used in a tensile test machine where load is applied until fracture occurs. This test is usually carried out in a **Universal testing machine** of the type shown in **clipping- 3.1.2.1**. The load at which the specimen finally ruptures is known as Ultimate load and the ratio of load to original cross-sectional area is the Ultimate stress.

3.1.2.1V SPACE FOR A UNIVERSAL TENSILE TEST CLIPPING

Similar tests are carried out for bending, shear and torsion and the results for different materials are available in handbooks. For design purpose an allowable stress is used in place of the critical stress to take into account the uncertainties including the following:

1) Uncertainty in loading.
2) Inhomogeneity of materials.
3) Various material behaviors. e.g. corrosion, plastic flow, creep.
4) Residual stresses due to different manufacturing process.

5) Fluctuating load (fatigue loading): Experimental results and plot- ultimate strength depends on number of cycles.
6) Safety and reliability.

For ductile materials, the yield strength and for brittle materials the ultimate strength are taken as the critical stress.

An allowable stress is set considerably lower than the ultimate strength. The ratio of ultimate to allowable load or stress is known as factor of safety i.e.

$$\frac{\text{Ultimate Stress}}{\text{Allowable Stress}} = \text{F.S.}$$

The ratio must always be greater than unity. It is easier to refer to the ratio of stresses since this applies to material properties.

3.1.3 Theories of failure

When a machine element is subjected to a system of complex stress system, it is important to predict the mode of failure so that the design methodology may be based on a particular failure criterion. Theories of failure are essentially a set of failure criteria developed for the ease of design.

In machine design an element is said to have failed if it ceases to perform its function. There are basically two types of mechanical failure:

(a) **Yielding**- This is due to excessive inelastic deformation rendering the machine part unsuitable to perform its function. This mostly occurs in ductile materials.

(b) **Fracture**- in this case the component tears apart in two or more parts. This mostly occurs in brittle materials.

There is no sharp line of demarcation between ductile and brittle materials. However a rough guideline is that if percentage elongation is less than 5% then the material may be treated as brittle and if it is more than 15% then the material is ductile. However, there are many instances when a ductile material may fail by fracture. This may occur if a material is subjected to

(a) Cyclic loading.

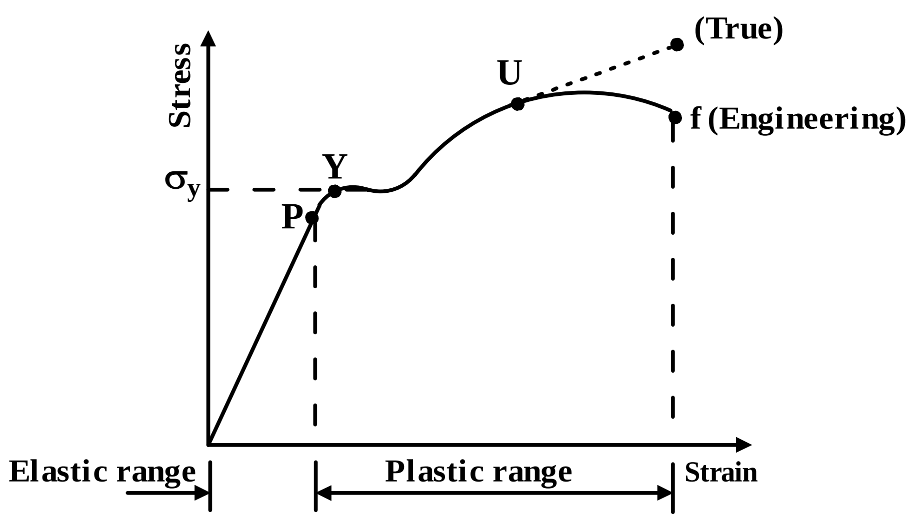
(b) Long term static loading at elevated temperature.

(c) Impact loading.

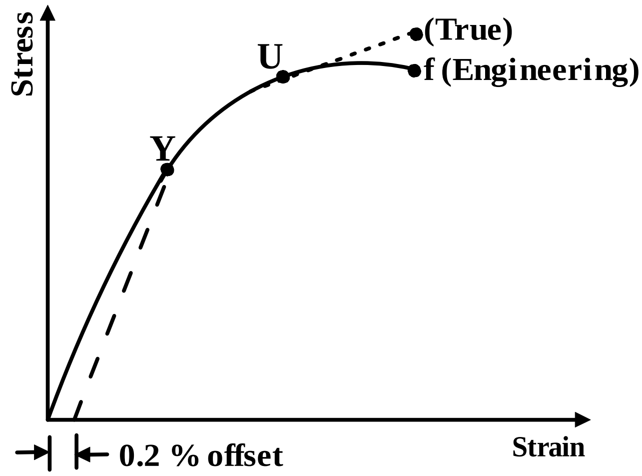
(d) Work hardening.

(e) Severe quenching.

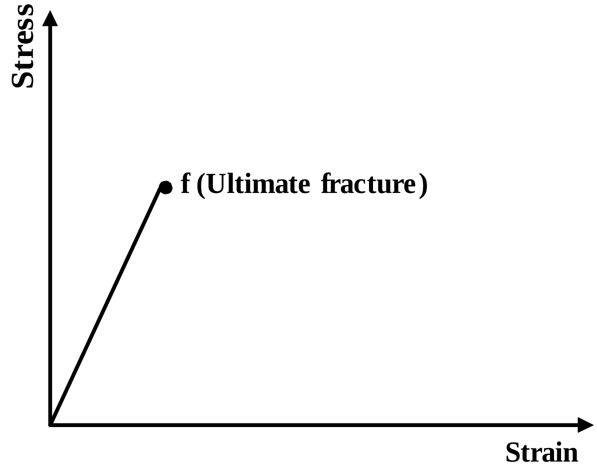
Yielding and fracture can be visualized in a typical tensile test as shown in the clipping- Typical engineering stress-strain relationship from simple tension tests for same engineering materials are shown in **figure- 3.1.3.1**.



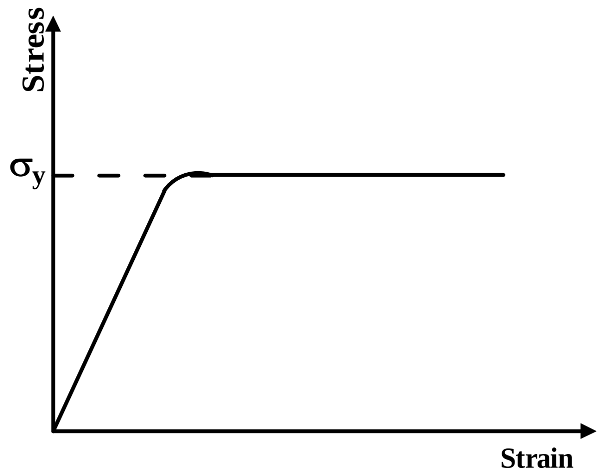
3.1.3.1F*- (a) Stress-strain diagram for a ductile material e.g. low carbon steel.*



3.1.3.1F*- (b) Stress-strain diagram for low ductility.*



3.1.3.1F*- (c) Stress-strain diagram for a brittle material.*



3.1.3.1F*- (d) Stress-strain diagram for an elastic – perfectly plastic material.*

For a typical ductile material as shown in **figure-3.1.3.1 (a)** there is a definite yield point where material begins to yield more rapidly without any change in stress level. Corresponding stress is σ_y . Close to yield point is the proportional limit which marks the transition from elastic to plastic range. Beyond elastic limit for an elastic- perfectly plastic material yielding would continue without further rise in stress i.e. stress-strain diagram would be parallel to parallel to strain axis beyond the yield point. However, for most ductile materials, such as, low-carbon steel beyond yield point the stress in the specimens rises upto a peak value known as ultimate tensile stress σ_o . Beyond this point the specimen starts to neck-down i.e. the reduction in cross-sectional area. However, the stress-strain curve falls till a point where fracture occurs. The drop in stress is apparent since original cross-sectional area is used to calculate the stress. If instantaneous cross-sectional area is used the curve would rise as shown in **figure- 3.1.3.1 (a)** . For a material with low ductility there is no definite yield point and usually off-set yield points are defined for convenience. This is shown in **figure-3.1.3.1**. For a brittle material stress increases linearly with strain till fracture occurs. These are demonstrated in the **clipping- 3.1.3.2** .

3.1.3.2V

SPACE FOR FATIGUE TEST CLIPPING

3.1.4 Yield criteria

There are numerous yield criteria, going as far back as Coulomb (1773). Many of these were originally developed for brittle materials but were later applied to ductile materials. Some of the more common ones will be discussed briefly here.

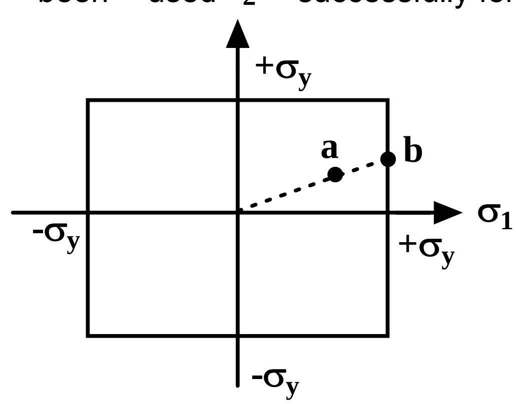
3.1.4.1 Maximum principal stress theory (Rankine theory)

According to this, if one of the principal stresses σ_1 (maximum principal stress), σ_2 (minimum principal stress) or σ_3 exceeds the yield stress, yielding would occur. In a two dimensional loading situation for a ductile material where tensile and compressive yield stress are nearly of same magnitude

$$\sigma_1 = \pm\, \sigma_y$$

$$\sigma_2 = \pm\sigma_y$$

Using this, a yield surface may be drawn, as shown in **figure- 3.1.4.1.1**. Yielding occurs when the state of stress is at the boundary of the rectangle. Consider, for example, the state of stress of a thin walled pressure vessel. Here $\sigma_1 = 2\sigma_2$, σ_1 being the circumferential or hoop stress and σ_2 the axial stress. As the pressure in the vessel increases the stress follows the dotted line. At a point (say) a, the stresses are still within the elastic limit but at b, σ_1 reaches σ_y although σ_2 is still less than σ_y. Yielding will then begin at point b. This theory of yielding has very poor agreement with experiment. However, the theory has been used successfully for brittle materials.



***3.1.4.1.1F**- Yield surface corresponding to maximum principal stress theory*

3.1.4.2 Maximum principal strain theory (St. Venant's theory)

According to this theory, yielding will occur when the maximum principal strain just exceeds the strain at the tensile yield point in either simple tension or compression. If ε_1 and ε_2 are maximum and minimum principal strains corresponding to σ_1 and σ_2, in the limiting case

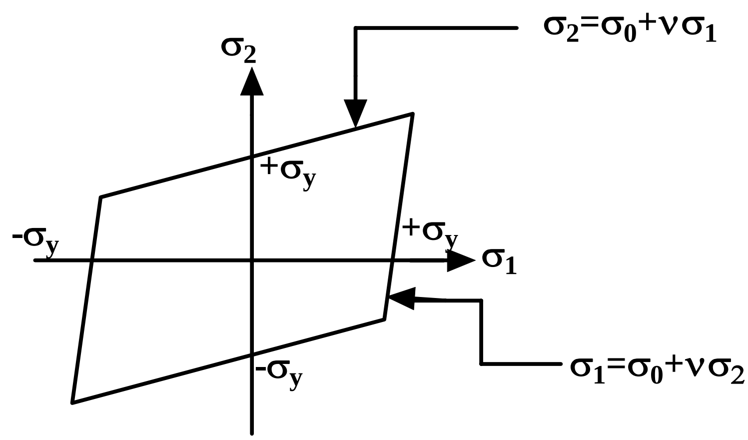
$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2) \qquad |\sigma_1| \geq |\sigma_2|$$

$$\varepsilon_2 = \frac{1}{E}(\sigma_2 - \nu\sigma_1) \qquad |\sigma_2| \geq |\sigma_1|$$

$$\text{This gives, } E\varepsilon_1 = \sigma_1 - \nu\sigma_2 = \pm\sigma_0$$

$$E\varepsilon_2 = \sigma_2 - \nu\sigma_1 = \pm\sigma_0$$

The boundary of a yield surface in this case is thus given as shown in **figure-3.1.4.2.1**

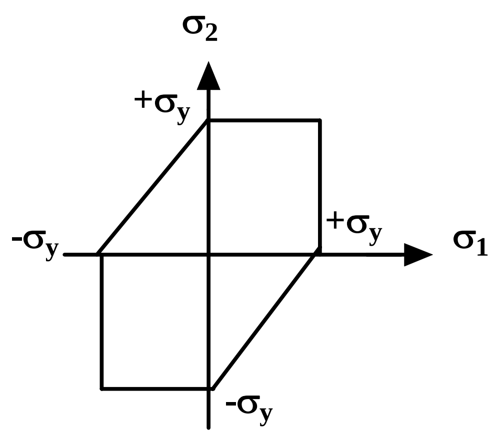


***3.1.4.2.1**- Yield surface corresponding to maximum principal strain theory*

3.1.4.3 Maximum shear stress theory (Tresca theory)

According to this theory, yielding would occur when the maximum shear stress just exceeds the shear stress at the tensile yield point. At the tensile yield point $\sigma_2 = \sigma_3 = 0$ and thus maximum shear stress is $\sigma_y/2$. This gives us six conditions for a three-dimensional stress situation:

$$\sigma_1 - \sigma_2 = \pm\sigma_y$$
$$\sigma_2 - \sigma_3 = \pm\sigma_y$$
$$\sigma_3 - \sigma_1 = \pm\sigma_y$$



***3.1.4.3.1F**- Yield surface corresponding to maximum shear stress theory*

In a biaxial stress situation (**figure-3.1.4.3.1**) case, $\sigma_3 = 0$ and this gives

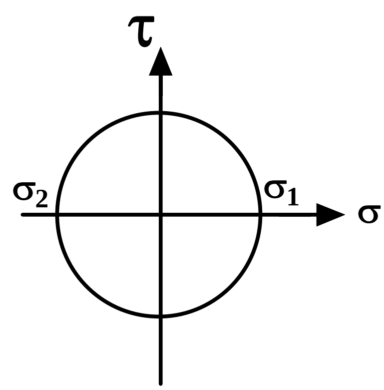
$$\sigma_1 - \sigma_2 = \sigma_y \quad \text{if } \sigma_1 > 0, \sigma_2 < 0$$
$$\sigma_1 - \sigma_2 = -\sigma_y \quad \text{if } \sigma_1 < 0, \sigma_2 > 0$$
$$\sigma_2 = \sigma_y \quad \text{if } \sigma_2 > \sigma_1 > 0$$
$$\sigma_1 = -\sigma_y \quad \text{if } \sigma_1 < \sigma_2 < 0$$
$$\sigma_1 = -\sigma_y \quad \text{if } \sigma_1 > \sigma_2 > 0$$
$$\sigma_2 = -\sigma_y \quad \text{if } \sigma_2 < \sigma_1 < 0$$

This criterion agrees well with experiment.

In the case of pure shear, $\sigma_1 = -\sigma_2 = k$ (say), $\sigma_3 = 0$

and this gives $\sigma_1 - \sigma_2 = 2k = \sigma_y$

This indicates that yield stress in pure shear is half the tensile yield stress and this is also seen in the Mohr's circle (figure- 3.1.4.3.2) for pure shear.



***3.1.4.3.2F**- Mohr's circle for pure shear*

3.1.4.4 Maximum strain energy theory (Beltrami's theory)

According to this theory failure would occur when the total strain energy absorbed at a point per unit volume exceeds the strain energy absorbed per unit volume at the tensile yield point. This may be given by

$$\frac{1}{2}\left(\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3\right) = \frac{1}{2}\sigma_y\varepsilon_y$$

$$\frac{1}{2}\left(\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3\right) = \frac{1}{2}\sigma_y\varepsilon_y$$

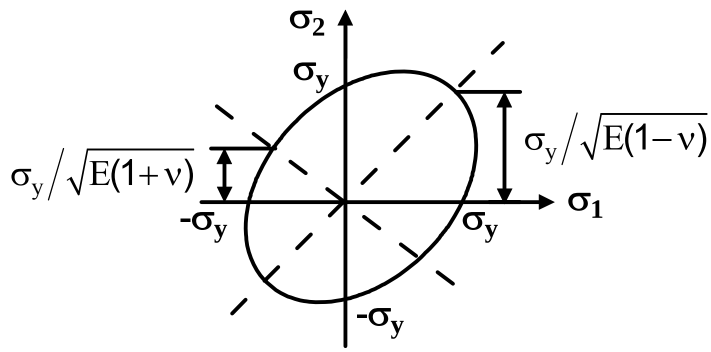
Substituting, ε_1, ε_2 , ε_3 and ε_y in terms of stresses we have

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\upsilon\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right) = \sigma_y^2$$

This may be written as

$$\left(\frac{\sigma_1}{\sigma_y}\right)^2 + \left(\frac{\sigma_2}{\sigma_y}\right)^2 - 2\nu\left(\frac{\sigma_1\sigma_2}{\sigma_y^2}\right) = 1$$

This is the equation of an ellipse and the yield surface is shown in **figure-3.1.4.4.1** .



3.1.4.4.1F*- Yield surface corresponding to Maximum strain energy theory***.**

It has been shown earlier that only distortion energy can cause yielding but in the above expression at sufficiently high hydrostatic pressure $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$ (say), yielding may also occur.

From the above we may write $\sigma^2(3-2\nu)=\sigma_y^2$ and if $\nu \sim 0.3$, at stress level lower than yield stress, yielding would occur. This is in contrast to the experimental as well as analytical conclusion and the theory is not appropriate.

3.1.4.5 Distortion energy theory(von Mises yield criterion)

According to this theory yielding would occur when total distortion energy absorbed per unit volume due to applied loads exceeds the distortion energy absorbed per unit volume at the tensile yield point. Total strain energy E_T and strain energy for volume change E_V can be given as

$$E_T = \frac{1}{2}\left(\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3\right) \text{ and } E_V = \frac{3}{2}\sigma_{av}\varepsilon_{av}$$

Substituting strains in terms of stresses the distortion energy can be given as

$E_d = E_T - E_V = \frac{2(1+\nu)}{6E}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1\right)$

At the tensile yield point, $\sigma_1 = \sigma_y$, $\sigma_2 = \sigma_3 = 0$ which gives

$$E_{dy} = \frac{2(1+\nu)}{6E}\sigma_y^2$$

The failure criterion is thus obtained by equating E_d and E_{dy}, which gives

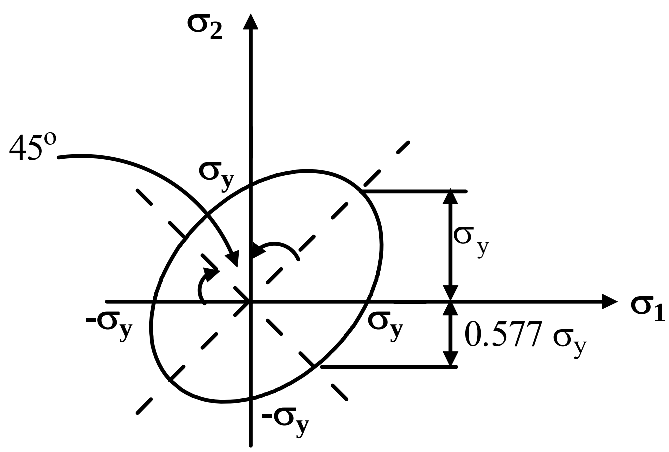
$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_y^2$$

In a 2-D situation if $\sigma_3 = 0$, the criterion reduces to

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = \sigma_y^2$$

$$\text{i.e.} \quad \left(\frac{\sigma_1}{\sigma_y}\right)^2 + \left(\frac{\sigma_2}{\sigma_y}\right)^2 - \left(\frac{\sigma_1}{\sigma_y}\right)\left(\frac{\sigma_2}{\sigma_y}\right) = 1$$

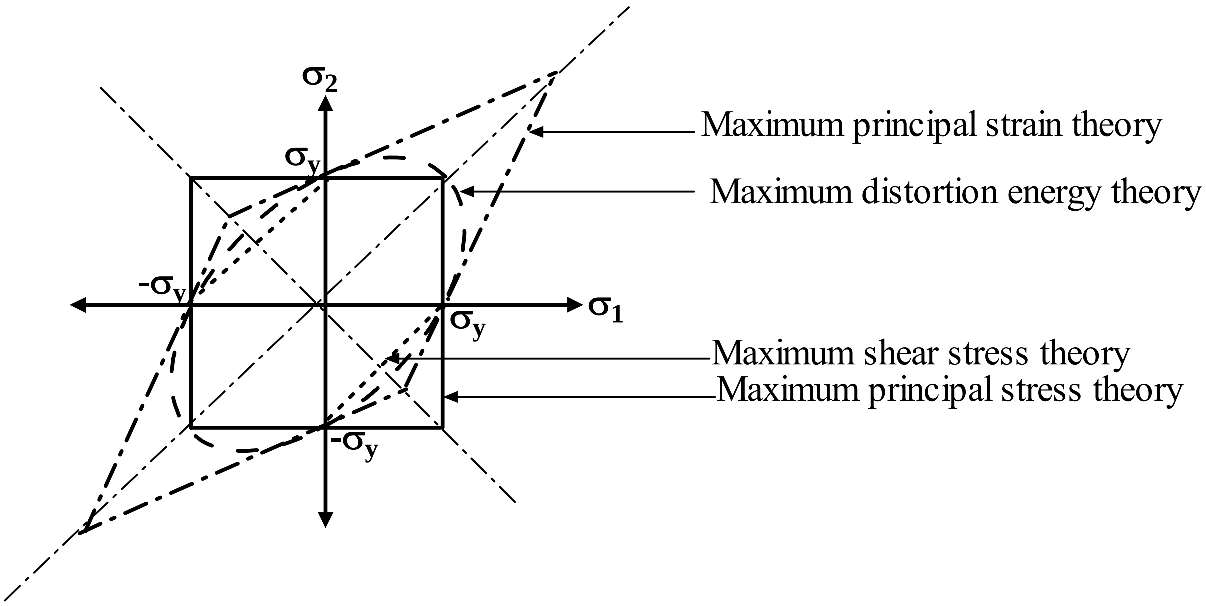
This is an equation of ellipse and the yield surface is shown in **figure-3.1.4.5.1** . This theory agrees very well with experimental results and is widely used for ductile materials.



3.1.4.5.1F*- Yield surface corresponding to von Mises yield criterion.*

3.1.5 Superposition of yield surface

A comparison among the different failure theories can be made by superposing the yield surfaces as shown in **figure- 3.1.5.1**.



***3.1.5.1F**- Comparison of different failure theories.*

It is clear that an immediate assessment of failure probability can be made just by plotting any experimental in the combined yield surface. Failure of ductile materials is most accurately governed by the distortion energy theory where as the maximum principal strain theory is used for brittle materials.

3.1.6 Problems with Answers

Q.1: A shaft is loaded by a torque of 5 KN-m. The material has a yield point of 350 MPa. Find the required diameter using

(a) Maximum shear stress theory

(b) Maximum distortion energy theory

Take a factor of safety of 2.5.

A.1:

Torsional shear stress induced in the shaft due to 5 KN-m torque is

$$\tau = \frac{16x(5x10^3)}{\pi d^3}$$ where d is the shaft diameter in m.

(b) Maximum shear stress theory,

$$\tau_{max} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2}$$

Since $\sigma_x = \sigma_y = 0$, τ_{max}=25.46x10^3/d^3 = $\frac{\sigma_Y}{2xF.S.} = \frac{350x10^6}{2x2.5}$

This gives d=71.3 mm.

(b) Maximum distortion energy theory

In this case σ_1 = 25.46x10^3/d^3

σ_2 = -25.46x10^3/d^3

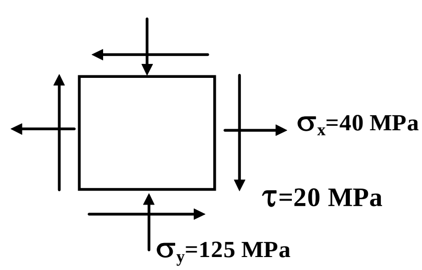
According to this theory,

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2 = 2(\sigma_Y / F.S)^2$$

Since σ_3 = 0, substituting values of σ_1 , σ_2 and σ_Y

D=68 mm.

Q.2: The state of stress at a point for a material is shown in the **figure-3.1.6.1**. Find the factor of safety using (a) Maximum shear stress theory (b) Maximum distortion energy theory. Take the tensile yield strength of the material as 400 MPa.



3.1.6.1F

A.2:

From the Mohr's circle, shown in **figure-3.1.6.2**

$\sigma_1 = 42.38$ MPa

$\sigma_2 = -127.38$ MPa

(a) Maximum shear stress theory

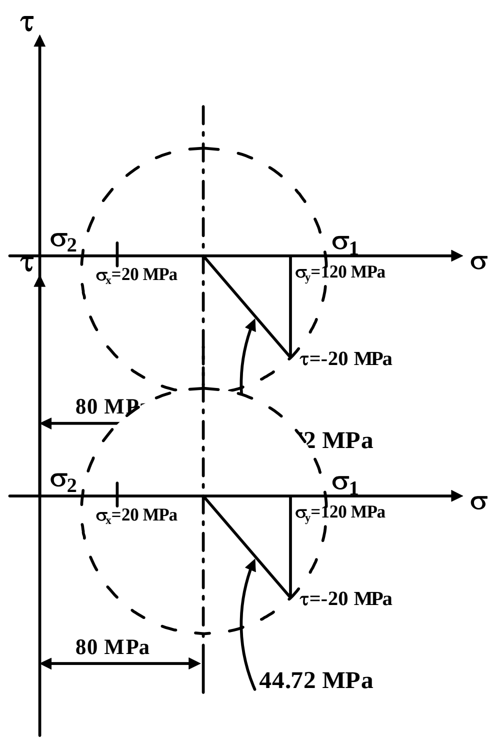
$$\frac{\sigma_1 - \sigma_2}{2} = \frac{\sigma_Y}{2xF.S}$$

This gives F.S = 2.356.

(b) Maximum distortion energy theory

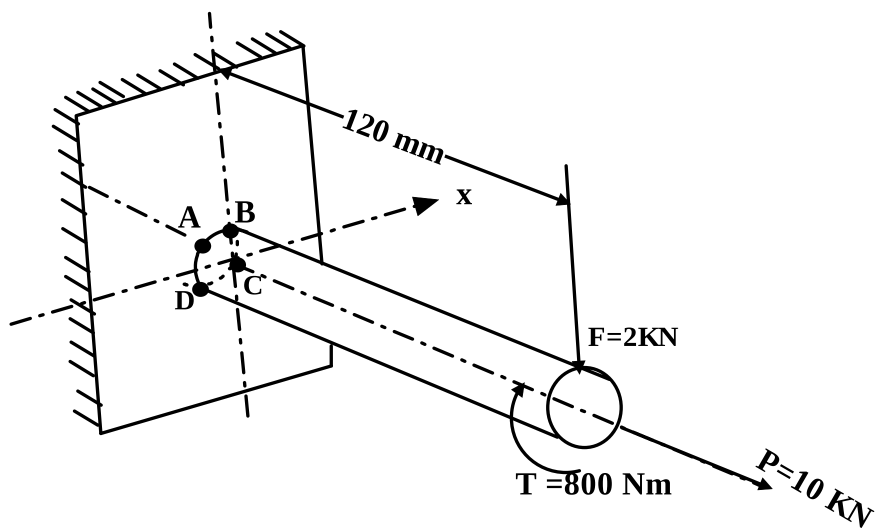
$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2 = 2(\sigma_Y / F.S)^2$$

If $\sigma_3 = 0$ this gives F.S = 2.613.



3.1.6.2F

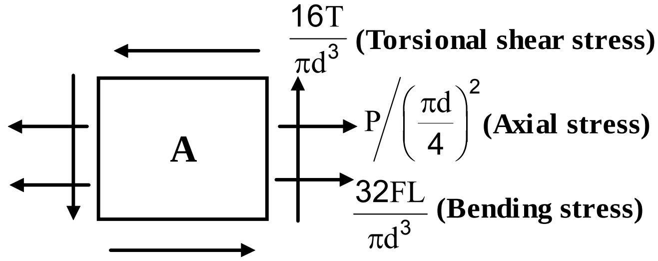
Q.3: A cantilever rod is loaded as shown in the **figure- 3.1.6.3**. If the tensile yield strength of the material is 300 MPa determine the rod diameter using (a) Maximum principal stress theory (b) Maximum shear stress theory (c) Maximum distortion energy theory.



3.1.6.3F

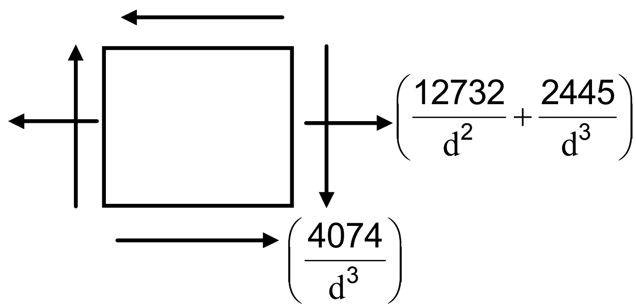
A.3:

At the outset it is necessary to identify the mostly stressed element. Torsional shear stress as well as axial normal stress is the same throughout the length of the rod but the bearing stress is largest at the welded end. Now among the four corner elements on the rod, the element A is mostly loaded as shown in **figure-3.1.6.4**



3.1.6.4F

Shear stress due to bending $\frac{VQ}{It}$ is also developed but this is neglected due to its small value compared to the other stresses. Substituting values of T, P, F and L, the elemental stresses may be shown as in **figure-3.1.6.5**:



3.1.6.5F

This gives the principal stress as

$$\sigma_{1,2}=\frac{1}{2}\left(\frac{12732}{d^2}+\frac{2445}{d^3}\right)\pm\sqrt{\frac{1}{4}\left(\frac{12732}{d^2}+\frac{2445}{d^3}\right)^2+\left(\frac{4074}{d^3}\right)^2}$$

(a) Maximum principal stress theory,

Setting $\sigma_1 = \sigma_Y$ we get d = 26.67 mm.

(b) Maximum shear stress theory,

Setting $\frac{\sigma_1-\sigma_2}{2}=\frac{\sigma_Y}{2}$, we get d = 30.63 mm.

(c) Maximum distortion energy theory,

Setting $(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_1-\sigma_3)^2=2(\sigma_Y)^2$

We get d = 29.36 mm.

3.1.7 Summary of this Lesson

Different types of loading and criterion for design of machine parts subjected to static loading based on different failure theories have been demonstrated. Development of yield surface and optimization of design criterion for ductile and brittle materials were illustrated.

Source:
http://nptel.ac.in/courses/Webcourse-contents/IIT%20Kharagpur/Machine%20design1/pdf/Module-3_lesson-1.pdf