Curvilinear Motion. Cartesian Coordinates

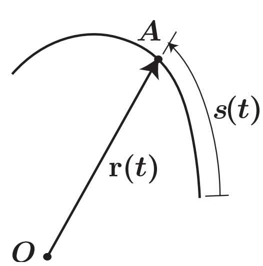
We will start by studying the motion of a *particle*. We think of a particle as a body which has mass, but has negligible dimensions. Treating bodies as particles is, of course, an idealization which involves an approximation. This approximation may be perfectly acceptable in some situations and not adequate in some other cases. For instance, if we want to study the motion of planets, it is common to consider each planet as a particle. This simplification is not adequate if we wish to study the precession of a gyroscope or a spinning top.

Kinematics of curvilinear motion

In *dynamics* we study the motion and the forces that cause, or are generated as a result of, the motion. Before we can explore these connections we will look first at the description of motion irrespective of the forces that produce them. This is the domain of *kinematics*. On the other hand, the connection between forces and motions is the domain of *kinetics* and will be the subject of the next lecture.

Position vector and Path

We consider the general situation of a particle moving in a three dimensional space. To locate the position of a particle in space we need to set up an origin point, O, whose location is known. The position of a particle A, at time t, can then be described in terms of the *position vector*, $\boldsymbol{r}$, joining points O and A. In general, this particle will not be still, but its position will change in time. Thus, the position vector will be a function of time, i.e. $\boldsymbol{r}(t)$. The curve in space described by the particle is called the *path*, or *trajectory*.



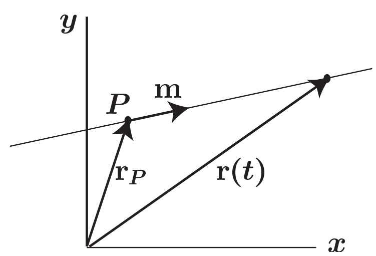
We introduce the *path* or *arc length coordinate*, s, which measures the distance traveled by the particle along the curved path. Note that for the particular case of rectilinear motion (considered in the review notes) the arc length coordinate and the coordinate, s, are the same.

Using the path coordinate we can obtain an alternative representation of the motion of the particle. Consider that we know $\boldsymbol{r}$ as a function of s, i.e. $\boldsymbol{r}(s)$, and that, in addition we know the value of the path coordinate as a function of time t, i.e. $s(t)$. We can then calculate the *speed* at which the particle moves on the path simply as $v = \dot{s} \equiv ds/dt$. We also compute the rate of change of speed as $a_t = \ddot{s} = d^2s/dt^2$.

We consider below some motion examples in which the position vector is referred to a fixed cartesian coordinate system.

Example — ***Motion along a straight line in 2D***

Consider for illustration purposes two particles that move along a line defined by a point P and a *unit* vector $\mathbf{m}$. We further assume that at $t = 0$, both particles are at point P. The position vector of the first particle is given by $\boldsymbol{r}_1(t) = \boldsymbol{r}_P + \mathbf{m}t = (r_{Px} + m_x t)\boldsymbol{i} + (r_{Py} + m_y t)\boldsymbol{j}$, whereas the position vector of the second particle is given by $\boldsymbol{r}_2(t) = \boldsymbol{r}_P + \mathbf{m}t^2 = (r_{Px} + m_x t^2)\boldsymbol{i} + (r_{Py} + m_y t^2)\boldsymbol{j}$.

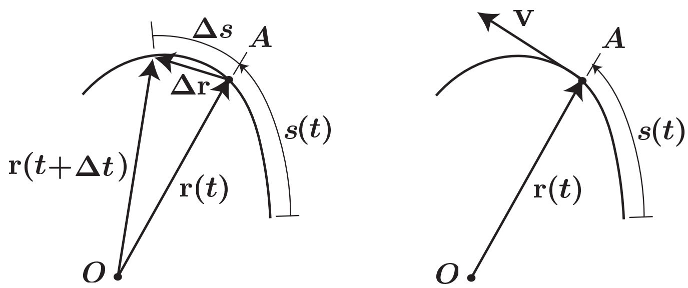


Clearly the path for these two particles is the same, but the speed at which each particle moves along the path is different. This is seen clearly if we parameterize the path with the path coordinate, s. That is, we write $\boldsymbol{r}(s) = \boldsymbol{r}_P + \mathbf{m}s = (r_{Px} + m_x s)\boldsymbol{i} + (r_{Py} + m_y s)\boldsymbol{j}$. It is straightforward to verify that s is indeed the path coordinate i.e. the distance between two points $\boldsymbol{r}(s)$ and $\boldsymbol{r}(s + \Delta s)$ is equal to Δs. The two motions introduced earlier simply correspond to two particles moving according to $s_1(t) = t$ and $s_2(t) = t^2$, respectively. Thus, $\boldsymbol{r}_1(t) = \boldsymbol{r}(s_1(t))$ and $\boldsymbol{r}_2(t) = \boldsymbol{r}(s_2(t))$.

It turns out that, in many situations, we will not have an expression for the path as a function of s. It is in fact possible to obtain the speed directly from $\boldsymbol{r}(t)$ without the need for an arc length parametrization of the trajectory.

Velocity Vector

We consider the positions of the particle at two different times t and $t + \Delta t$, where Δt is a small increment of time. Let $\Delta \boldsymbol{r} = \boldsymbol{r}(r + \Delta t) - \boldsymbol{r}(t)$, be the displacement vector as shown in the diagram.



The average velocity of the particle over this small increment of time is

$$\boldsymbol{v}_{ave} = \frac{\Delta \boldsymbol{r}}{\Delta t} ,$$

which is a vector whose direction is that of $\Delta \boldsymbol{r}$ and whose magnitude is the length of $\Delta \boldsymbol{r}$ divided by Δt. If Δt is small, then $\Delta \boldsymbol{r}$ will become tangent to the path, and the modulus of $\Delta \boldsymbol{r}$ will be equal to the distance the particle has moved on the curve Δs.

The instantaneous velocity vector is given by

$$\boldsymbol{v} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{r}}{\Delta t} \equiv \frac{d\boldsymbol{r}(t)}{dt} \equiv \dot{\boldsymbol{r}} , \tag{1}$$

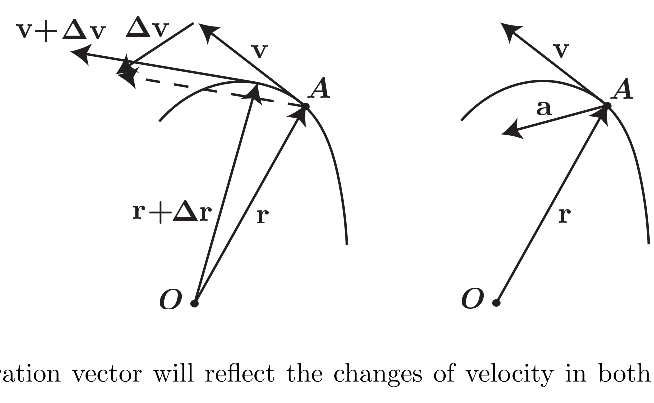
and is *always* tangent to the path. The magnitude, or speed, is given by

$$v = |\boldsymbol{v}| = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} \equiv \frac{ds}{dt} \equiv \dot{s} .$$

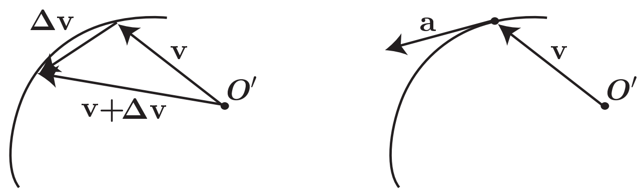
Acceleration Vector

In an analogous manner, we can define the acceleration vector. Particle A at time t, occupies position $\boldsymbol{r}(t)$, and has a velocity $\boldsymbol{v}(t)$, and, at time $t + \Delta t$, it has position $\boldsymbol{r}(t + \Delta t) = \boldsymbol{r}(t) + \Delta \boldsymbol{r}$, and velocity $\boldsymbol{v}(t + \Delta t) = \boldsymbol{v}(t) + \Delta \boldsymbol{v}$. Considering an infinitesimal time increment, we define the acceleration vector as the derivative of the velocity vector with respect to time,

$$\boldsymbol{a} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{v}}{\Delta t} \equiv \frac{d\boldsymbol{v}}{dt} = \frac{d^2\boldsymbol{r}}{dt^2} . \tag{2}$$



We note that the acceleration vector will reflect the changes of velocity in both magnitude and direction. The acceleration vector will, in general, not be tangent to the trajectory (in fact it is only tangent when the velocity vector does not change direction). A sometimes useful way to visualize the acceleration vector is to translate the velocity vectors, at different times, such that they all have a common origin, say, O'. Then, the heads of the velocity vector will change in time and describe a curve in space called the *hodograph*. We then see that the acceleration vector is, in fact, tangent to the hodograph at every point.



Expressions (1) and (2) introduce the concept of *derivative of a vector*. Because a vector has both magnitude and direction, the derivative will be non-zero when either of them changes **(see the review notes on vectors)**. In general, the derivative of a vector will have a component which is *parallel* to the vector itself, and is *due to the magnitude change*; and a component which is *orthogonal* to it, and is *due to the direction change*.

Note — ***Unit tangent and arc-length parametrization***

The unit tangent vector to the curve can be simply calculated as

$$\boldsymbol{e}_t = \boldsymbol{v}/v.$$

It is clear that the tangent vector depends solely on the geometry of the trajectory and not on the speed at which the particle moves along the trajectory. That is, the geometry of the trajectory determines the tangent vector, and hence the direction of the velocity vector. How fast the particle moves along the trajectory determines the magnitude of the velocity vector. This is clearly seen if we consider the arc-length parametrization of the trajectory $\boldsymbol{r}(s)$. Then, applying the chain rule for differentiation, we have that,

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{r}}{ds}\frac{ds}{dt} = \boldsymbol{e}_t v \ ,$$

where, $\dot{s} = v$, and we observe that $d\boldsymbol{r}/ds = \boldsymbol{e}_t$. The fact that the modulus of $d\boldsymbol{r}/ds$ is always unity indicates that the distance traveled, along the path, by $\boldsymbol{r}(s)$, (recall that this distance is measured by the coordinate s), per unit of s is, in fact, unity!. This is not surprising since by definition the distance between two neighboring points is ds, i.e. $|d\boldsymbol{r}| = ds$.

Cartesian Coordinates

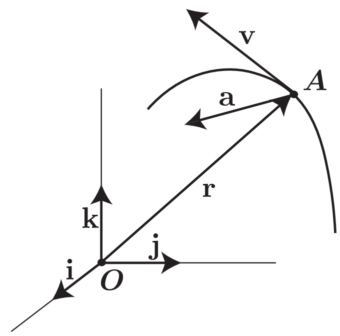
When working with fixed cartesian coordinates, vector differentiation takes a particularly simple form. Since the vectors $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ do not change, the derivative of a vector $\boldsymbol{A}(t) = A_x(t)\boldsymbol{i} + A_y(t)\boldsymbol{j} + A_z(t)\boldsymbol{k}$, is simply $\dot{\boldsymbol{A}}(t) = \dot{A}_x(t)\boldsymbol{i} + \dot{A}_y(t)\boldsymbol{j} + \dot{A}_z(t)\boldsymbol{k}$. That is, the components of the derivative vector are simply the derivatives of the components.

Thus, if we refer the position, velocity, and acceleration vectors to a fixed cartesian coordinate system, we have,

$$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k} \tag{3}$$

$$\boldsymbol{v}(t) = v_x(t)\boldsymbol{i} + v_y(t)\boldsymbol{j} + v_z(t)\boldsymbol{k} = \dot{x}(t)\boldsymbol{i} + \dot{y}(t)\boldsymbol{j} + \dot{z}(t)\boldsymbol{k} = \dot{\boldsymbol{r}}(t) \tag{4}$$

$$\boldsymbol{a}(t) = a_x(t)\boldsymbol{i} + a_y(t)\boldsymbol{j} + a_z(t)\boldsymbol{k} = \dot{v}_x(t)\boldsymbol{i} + \dot{v}_y(t)\boldsymbol{j} + \dot{v}_z(t)\boldsymbol{k} = \dot{\boldsymbol{v}}(t) \tag{5}$$

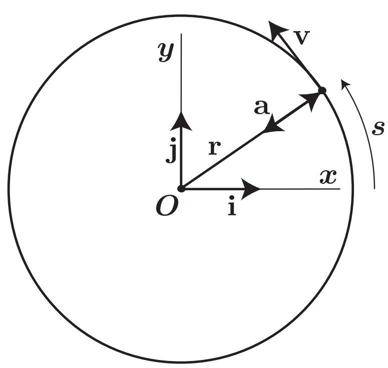


Here, the speed is given by $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$, and the magnitude of the acceleration is $a = \sqrt{a_x^2 + a_y^2 + a_z^2}$. The advantages of cartesian coordinate systems is that they are simple to use, and that if $\boldsymbol{a}$ is constant, or a function of time only, we can integrate each component of the acceleration and velocity independently as shown in the ballistic motion example.

Example ***Circular Motion***

We consider motion of a particle along a circle of radius R at a constant speed v_0. The parametrization of a circle in terms of the arc length is

$$\boldsymbol{r}(s) = R\cos(\frac{s}{R})\boldsymbol{i} + R\sin(\frac{s}{R})\boldsymbol{j} \ .$$



Since we have a constant speed v_0, we have $s = v_0 t$. Thus,

$$\boldsymbol{r}(t) = R\cos(\frac{v_0 t}{R})\boldsymbol{i} + R\sin(\frac{v_0 t}{R})\boldsymbol{j} \ .$$

The velocity is

$$\boldsymbol{v}(t) = \frac{d\boldsymbol{r}(t)}{dt} = -v_0\sin(\frac{v_0 t}{R})\boldsymbol{i} + v_0\cos(\frac{v_0 t}{R})\boldsymbol{j} \ ,$$

which, clearly, has a constant magnitude $|\boldsymbol{v}| = v_0$. The acceleration is,

$$\boldsymbol{a}(t) = \frac{d\boldsymbol{r}(t)}{dt} = -\frac{v_0^2}{R}\cos(\frac{v_0 t}{R})\boldsymbol{i} - \frac{v_0^2}{R}\sin(\frac{v_0 t}{R})\boldsymbol{j} \ .$$

Note that, the acceleration is perpendicular to the path (in this case it is parallel to $\boldsymbol{r}$), since the velocity vector changes direction, but not magnitude.

We can also verify that, from $\boldsymbol{r}(s)$, the unit tangent vector, $\boldsymbol{e}_t$, could be computed directly as

$$\boldsymbol{e}_t = \frac{d\boldsymbol{r}(s)}{ds} = -\sin(\frac{s}{R})\boldsymbol{i} + \cos(\frac{s}{R}) = -\sin(\frac{v_0 t}{R})\boldsymbol{i} + \cos(\frac{v_0 t}{R})\boldsymbol{j} \ .$$

Example ***Motion along a helix***

The equation $\boldsymbol{r}(t) = R\cos t\boldsymbol{i} + R\sin t\boldsymbol{j} + ht\boldsymbol{k}$, defines the motion of a particle moving on a helix of radius R, and pitch $2\pi h$, at a constant speed. The velocity vector is given by

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = -R\sin t\boldsymbol{i} + R\cos t\boldsymbol{j} + h\boldsymbol{k} \ ,$$

and the acceleration vector is given by,

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = -R\cos t\boldsymbol{i} + -R\sin t\boldsymbol{j} \ .$$

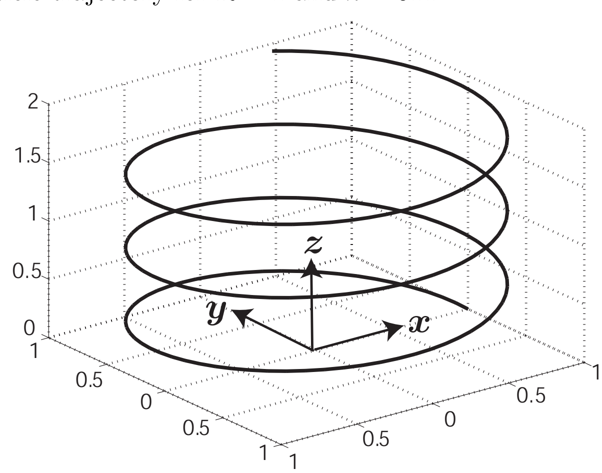
In order to determine the speed at which the particle moves we simply compute the modulus of the velocity vector,

$$v = |\boldsymbol{v}| = \sqrt{R^2\sin^2 t + R^2\cos^2 t + h^2} = \sqrt{R^2 + h^2} \ .$$

If we want to obtain the equation of the path in terms of the arc-length coordinate we simply write,

$$ds = |d\boldsymbol{r}| = vdt = \sqrt{R^2 + h^2}\ dt \ .$$

Integrating, we obtain $s = s_0 + \sqrt{R^2 + h^2}\ t$, where s_0 corresponds to the path coordinate of the particle at time zero. Substituting t in terms of s, we obtain the expression for the position vector in terms of the arc-length coordinate. In this case, $\boldsymbol{r}(s) = R\cos(s/\sqrt{R^2 + h^2})\boldsymbol{i} + R\sin(s/\sqrt{R^2 + h^2})\boldsymbol{j} + hs/\sqrt{R^2 + h^2}\boldsymbol{k}$. The figure below shows the particle trajectory for $R = 1$ and $h = 0.1$.



Example **_Ballistic Motion_**

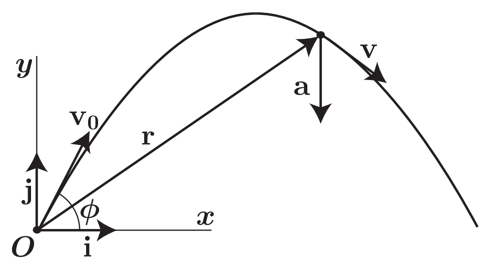
Consider the free-flight motion of a projectile which is initially launched with a velocity $\boldsymbol{v}_0 = v_0 \cos\phi \boldsymbol{i} + v_0 \sin\phi \boldsymbol{j}$. If we neglect air resistance, the only force on the projectile is the weight, which causes the projectile to have a constant acceleration $\boldsymbol{a} = -g\boldsymbol{j}$. In component form this equation can be written as $dv_x/dt = 0$ and $dv_y/dt = -g$. Integrating and imposing initial conditions, we get

$$v_x = v_0 \cos\phi, \qquad v_y = v_0 \sin\phi - gt \ ,$$

where we note that the horizontal velocity is constant. A further integration yields the trajectory

$$x = x_0 + (v_0 \cos\phi)\ t, \qquad y = y_0 + (v_0 \sin\phi)\ t - \frac{1}{2} g t^2 \ ,$$

which we recognize as the equation of a parabola.



The maximum height, y_{mh}, occurs when $v_y(t_{mh}) = 0$, which gives $t_{mh} = (v_0/g)\sin\phi$, or,

$$y_{mh} = y_0 + \frac{v_0^2 \sin^2\phi}{2g} \ .$$

The range, x_r, can be obtained by setting $y = y_0$, which gives $t_r = (2v_0/g)\sin\phi$, or,

$$x_r = x_0 + \frac{2v_0^2 \sin\phi \cos\phi}{g} = x_0 + \frac{v_0^2 \sin(2\phi)}{g} \ .$$

We see that if we want to maximize the range x_r, for a given velocity v_0, then $\sin(2\phi) = 1$, or $\phi = 45^o$. Finally, we note that if we want to model a more realistic situation and include aerodynamic drag forces of the form, say, $-\kappa v^2$, then we would not be able to solve for x and y independently, and this would make the problem considerably more complicated (usually requiring numerical integration).

ADDITIONAL READING

J.L. Meriam and L.G. Kraige, *Engineering Mechanics, DYNAMICS*, 5th Edition

2/1, 2/3, 2/4