Conservation of Momentum using Control Volumes

Conservation of Linear Momentum

Recall the conservation of linear momentum law for a system:

$$\frac{d}{dt}(m\mathbf{V})_{sys} = \sum \mathbf{F}_{sys}$$

In order to convert this for use in a control volume, use RTT with B = m**V**, beta = **V**

we get:

$$\frac{d}{dt}(m\mathbf{V})_{sys} = \sum \mathbf{F}_{sys} = \sum \mathbf{F}_{cv} = \frac{d}{dt}\int_{cv} \rho \cdot dv + \oint_{cs} \rho \cdot \mathbf{V}(\mathbf{V} \bullet \mathbf{n})dA$$

NOTE: Recall that at any instant of time t, the system & CV occupy the SAME physical space.

So, the forces of the system are the same at the forces of the control volume at a given instant.

- For a **fixed control volume** we have the following equation:

$$\sum \vec{F}_{cv} = \frac{d}{dt}\int_{cv} \rho \underline{V} d\forall + \oint_{cs} \rho \underline{V}(\underline{V} \bullet \underline{n})dA$$

  This is a vector equation so it has three components.

  - First, let us consider the component in the **<u>X-direction</u>**. We will drop the cv subscript since it is understood. The conservation of linear momentum equation becomes:

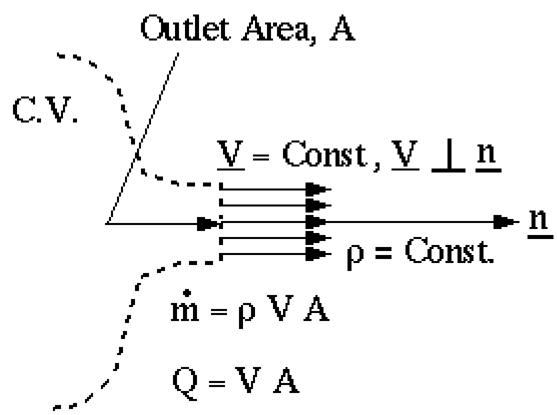
$$\sum F_X = \frac{d}{dt}\int_{cv} \rho u d\forall + \oint_{cs} \rho u(\underline{V} \bullet \underline{n})dA$$

    Notice that the <u>V</u> dot <u>n</u> term is a scalar, not a vector.

  - Next, let us consider the component in the **<u>Y-direction</u>**. The conservation of linear momentum applied to the y-direction becomes:

$$\sum F_Y = \frac{d}{dt}\int_{cv} \rho v d\forall + \oint_{cs} \rho v(\underline{V} \bullet \underline{n})dA$$

- Similarly, the conservation of momentum could be applied to the **Z-direction**.
- It is now time for a few simplifications for the right hand side of the CLM equation.
  1. First, let us assume that we have one dimensional inlets and outlets. This implies that our velocity vector $\underline{V}$ is parallel to our normal to the surface vector $\underline{n}$. We also assume that the velocity is constant across the inlet or outlet surface.

     Assuming density is constant, we can rewrite the last term in our **CLM** equation:

     $$\oint_{cs} \rho \underline{V}(\underline{V} \bullet \underline{n})\,dA = \sum_{out} \rho V A \underline{V} - \sum_{in} \rho V A \underline{V}$$

     This holds true because $\underline{V}$ dot $\underline{n}$ = V for outlets and -V for inlets. We can make a further simplification if we notice that the definition for mass flow rate is (density)(velocity)(area). Therefore our equation for the flux term becomes:

     $$\sum_{out} \dot{m}\vec{V} - \sum_{in} \dot{m}\vec{V}$$

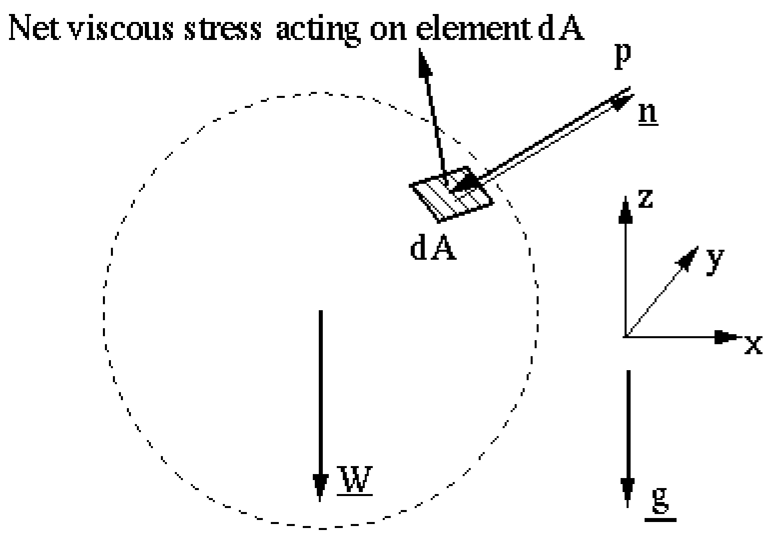
  2. If the flow is **steady** we can drop the (d/dt) term. Most of the problems we deal with having steady flow also have one dimensional inlets and outlets with constant density. Our equation for conservation of linear momentum now becomes:

     $$\sum \vec{F}_{CV} = \sum_{out} \dot{m}\vec{V} - \sum_{in} \dot{m}\vec{V}$$

Notice that this is a vector equation. Therefore we can break this up into three components. Since the velocity vector = (u,v,w) and the force vector = (F_x,F_y,F_z), our equation can be rewritten into three equations:

$$\sum F_x = \sum_{out} \dot{m}u - \sum_{in} \dot{m}u$$
$$\sum F_y = \sum_{out} \dot{m}v - \sum_{in} \dot{m}v$$
$$\sum F_z = \sum_{out} \dot{m}w - \sum_{in} \dot{m}w$$

- Now lets return to the left side of the CLM equation (the Force term). This term represents the sum of all the forces acting on the control volume. There are several types of forces that can act on our control volume.



  - Body Forces:

$$\sum \underline{F}_{grav} = -W\,\underline{k}$$

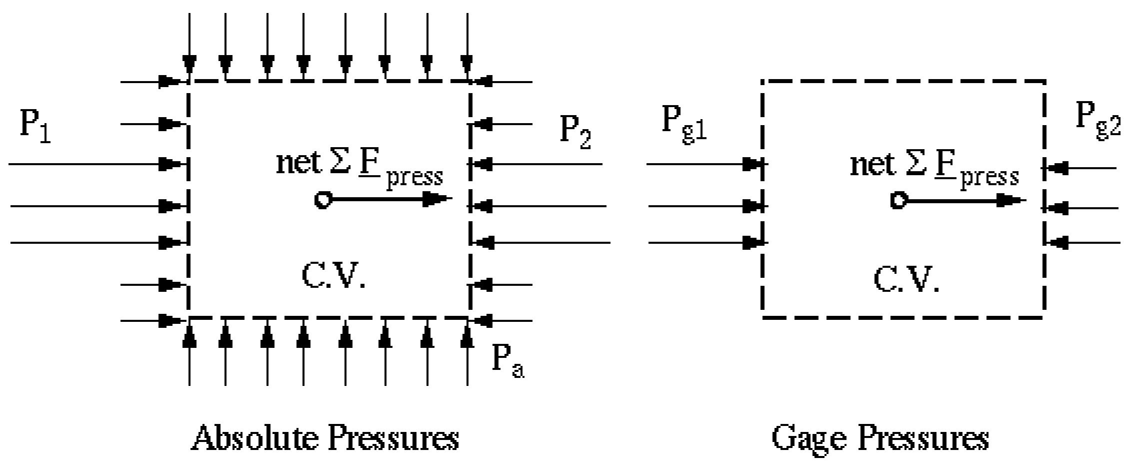
Gravity is a body force that acts in the negative z-direction.

  - Surface Forces: These forces include:
    - Pressure-contributes only normal forces.

   - Viscous-viscous and frictional forces contribute normal and shear forces.
   - Other-forces due to control volume cutting through bolts or struts.
- Let us revisit the **pressure force**. Recall that P is always defined as <u>positive inward</u> and <u>n</u> is always defined as <u>positive outward</u>.. These two directions are opposite. Therefore it turns out that

$$\sum \underline{F}_{press} = -\oint_{cs} P \, \underline{n} \, dA$$

  You can use either **absolute or gage** pressure in this equation as long as you are consistent everywhere! To prove this, consider an arbitrary control volume with a certain pressure field. Suppose there is a high pressure at two locations and atmospheric pressure everywhere else. Realizing the definition of gage pressure, Pg = Pabsolute - Patmospheric, we can subtract atmospheric pressure from everything. We realize that the sum of the forces due to pressure is the same in both cases.
  The **net** pressure force <u>will not change</u>. This holds true because any shape in a uniform pressure field has zero net pressure force.



- Now let us return to the **viscous forces**.

$$\sum \underline{F}_{visc} = difficult$$

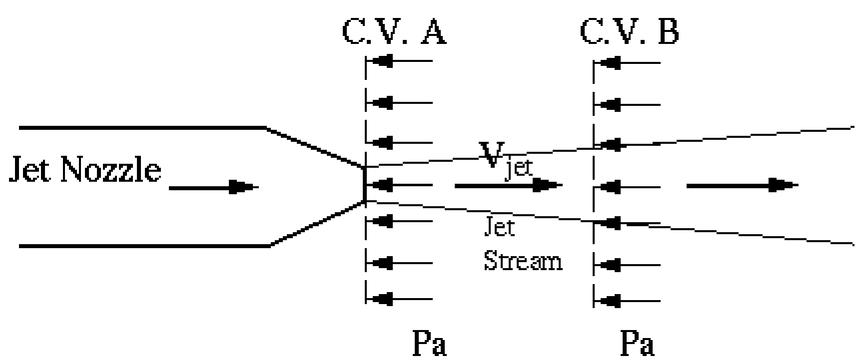
  This force is difficult to calculate because it has a normal and a tangential component acting in an arbitrary direction. In order to solve for this term you would have to integrate the shear stress over the entire control surface. Fortunately we usually do not have to integrate for this term. We can often choose a "wise" control volume to eliminate this difficult term. This is done by picking a control volume so that no part of the control surface has viscous forces acting on it. We could also keep the viscous term as are unknown term and solve for it by computing our other terms.

- Finally let us consider **other surface forces**. These forces include whatever is left. In other words, any force acting on the control volume that is not accounted for. Some examples of these types of forces include struts, bolts, cables, ropes in tension, forces holding a control volume in place, etc.
- The final form of the momentum equation for steady flow with one dimensional inlets and outlets is:

$$\sum \vec{F}_{CV} = \sum \vec{F}_{body} + \sum \vec{F}_{press} + \sum \vec{F}_{visc} + \sum \vec{F}_{other} = \sum_{out} \dot{m}\vec{V} - \sum_{in} \dot{m}\vec{V}$$

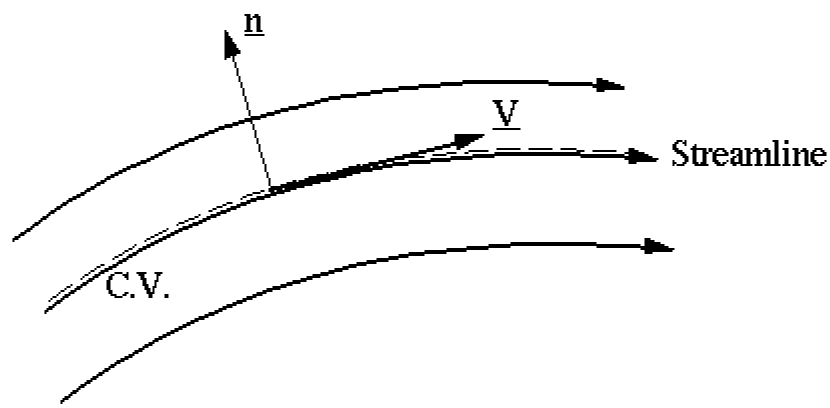
Here are some rules when using the momentum equation.

- An incompressible jet has an exit pressure equal to the ambient pressure. The ambient pressure is usually atmospheric. Therefore, when you draw your control volume, it is wise to slice through the jet at the exit plane. This will cause the pressure to be ambient everywhere.



- The pressure is approximately ambient in any slice through an incompressible jet. As you move down the stream, the pressure is still equal to the ambient pressure.

- The mass flow rate and momentum flow rate equal zero across a streamline. Therefore it is often advantageous to pick a control surface so as to run along a streamline.



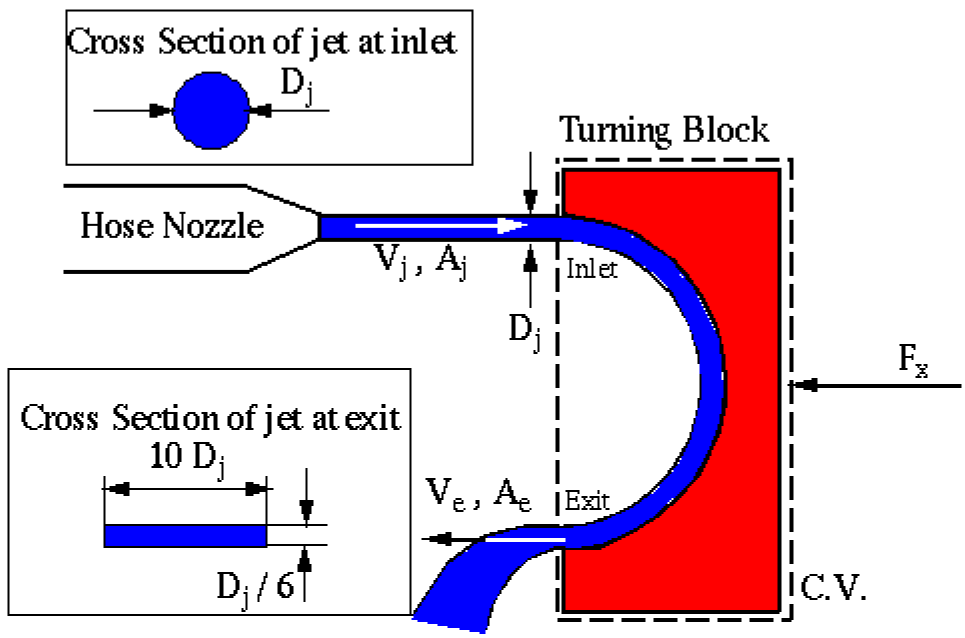
When you do this V dot n is equal to zero because they are perpendicular. This will cause the flux terms(mass, momentum, and energy) to be zero along the streamlines.

- Mass flow rate and momentum flow rate equal zero along a solid wall. This is true since a solid wall is always a streamline. No mass can go through a solid wall.

Note that the pressure and viscous force terms may not be zero along streamlines, however sometimes it is "wise" to pick a control volume along a solid wall.

Example Problems

Given: A water jet of velocity V_j and thickness D_j impinges on a turning block which is held in place by a force F_x, as shown in the sketch.



As the water leaves the block, the round jet flattens out and slows down due to friction along the wall. The water turns a full 180 degrees and flattens into a rectangle shape of thickness $D_j/6$ and width $10D_j$ in cross section. The flow is steady.

a) **Find**: V_e, the exit velocity of the jet.

Solution:

- First, pick a control volume which cuts through the inlet and exit, but does not include the turning block.
- Use conservation of mass for steady 1-D flow:

$$\sum_{in} \dot{m} = \sum_{out} \dot{m}$$

$$\rho V_j A_j = \rho V_e A_e$$

$$V_j \frac{\pi}{4} D_j^2 = V_e (10D_j)(\frac{1}{6} D_j)$$

$$V_e = \frac{3\pi}{20} V_j$$

b) **Find**: F_x, the force required to hold block in place.

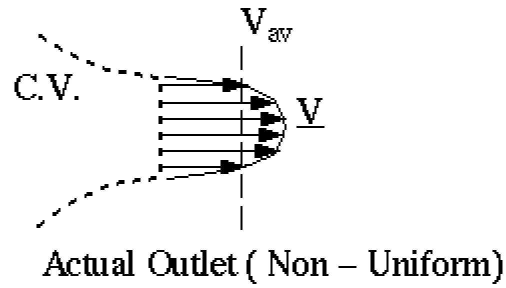
Solution:

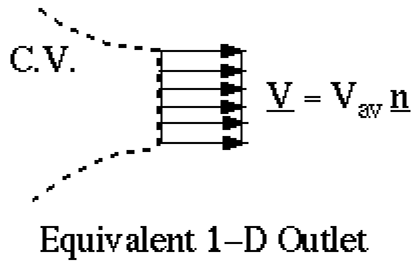
- Choose a new control volume because friction along the wall would be very hard to evaluate with the previous C.V. Pick a C.V. which cuts through the inlet and outlet and that includes the whole turning block. The wisest choice of control volume is one which cuts through where the force F_x acts on the turning block, since F_x is our unknown.

- Write momentum equation for steady 1-D inlet/outlet in the x-direction:
$$\sum F_x = \sum F_{x_{grav}} + \sum F_{x_{press}} + \sum F_{x_{visc}} + \sum F_{x_{other}} = \sum_{out} \dot{m}u - \sum_{in} \dot{m}u$$
Examine each term on the left: $\sum F_{x_{grav}} = 0$ (no grav in x-dir), $\sum F_{x_{press}} = 0$ (p = p_a everywhere), $\sum F_{x_{visc}} = 0$ (wise choice of C.V), and $\sum F_{x_{other}} = -F_x$
- Thus, the momentum equation reduces to
$$-F_x = \dot{m}(-V_e) - \dot{m}(V_j)$$
- Now, recall, $\dot{m} = \rho VA$ at an inlet or exit. Here at the inlet,
$$\dot{m} = \rho V_j \frac{\pi}{4} D_j^2 = \dot{m}_{in} = \dot{m}_{out}$$
- So, solve for F_x:
$$F_x = \dot{m}V_j + \dot{m}V_e$$
Or, finally, $F_x = \rho V_j \frac{\pi}{4} D_j^2 \left(V_j + \frac{3\pi}{20} V_j \right)$, which is our final answer.
- Note: This is not merely an academic exercise. A testing facility at NASA Langley Research Center uses this principle to propel a test rig along a track to test airplane landing gear, etc. The experimental facility, called the Aircraft Landing Dynamics Facility, uses a high pressure water jet that hits a turning bucket on the test cart, much like in the above problem. The cart is propelled from zero to 250 miles per hour in two seconds flat!

Momentum Flux Correction Factor

- Again, just like in the conservation of mass equation, even if we don't really have one-dimensional (uniform) inlets and outlets, we would still like to use the simplified version of the conservation of momentum equation. I.e. we would like to use V_{av} instead of V in the equation.





- The equivalent profile and actual profile thus have identical mass flow rates, $\dot{m}$.
- What about momentum flow rate, or momentum flux, MF? Do actual and equivalent profiles have the same momentum flux? ***No***! It turns out, after integration, that:
  $MF_{actual} > MF_{equiv}$
- So, we can't just substitute V_{av} for V in the momentum equation or we will get the wrong answer. So, instead, let's introduce a momentum flux correction factor, β:
  Define: $\beta = \frac{MF_{actual}}{MF_{equiv}}$
  It turns out that: $\beta = \frac{1}{A}\int_A \left(\frac{u}{V_{av}}\right)^2 dA$,
  and, $\beta \text{ always } \geq 1$.
- Examples:
  1) 1-D uniform flow, $\beta = 1$ by definition
  2) fully developed laminar pipe flow, $\beta = \frac{4}{3}$
  3) fully developed turbulent pipe flow, $\beta \approx 1.02$
- Now, we can use the one-dimensional form of the momentum equation, but with these momentum flux correction factors thrown in:
  For a fixed control volume with steady flow,
  $\sum F = \sum_{out} \dot{m} V_{av} \beta - \sum_{in} \dot{m} V_{av} \beta$, and $\dot{m} = \rho V_{av} A$

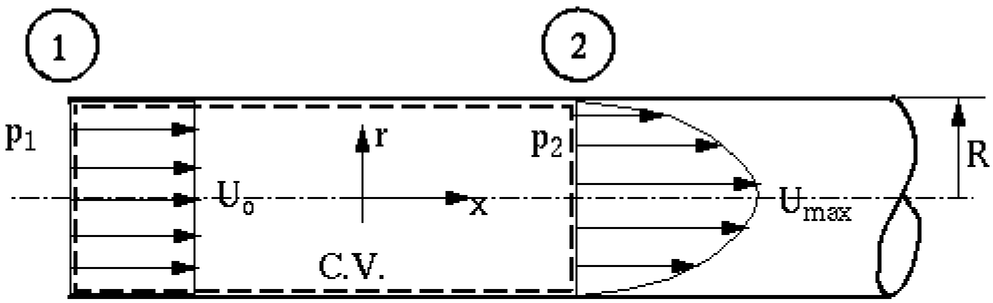
- Fortunately, most problems in real life applications are turbulent, not laminar, and we can usually neglect the momentum flux correction factors since $\beta_{turb} \approx 1.02$ is close to 1.0.

Most Useful Form of the Momentum Equation

- For steady flow with a fixed control volume, the most useful form of the momentum equation is thus:
$$\sum \vec{F} = \sum \vec{F}_{grav} + \sum \vec{F}_{press} + \sum \vec{F}_{visc} + \sum \vec{F}_{other} = \sum_{out} \dot{m}\vec{V}_{av}\beta - \sum_{in} \dot{m}\vec{V}_{av}\beta$$
, where β = momentum flux correction factor, and $\dot{m} = \rho V_{av} A$.

Example Problem

Given: Consider incompressible flow in the entrance of a circular tube.



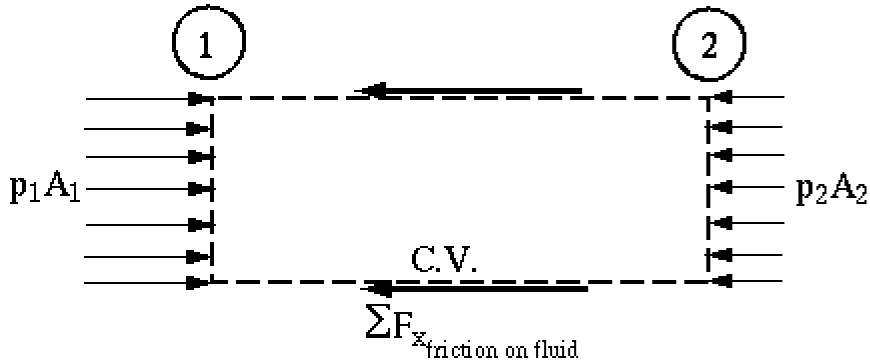
The inlet flow is uniform $u_1 = U_0$. The flow at section 2 is developed laminar pipe flow. U_0, p_1, p_2, R, and ρ, are also known. At section 2, $u = u_{max}\left(1 - \frac{r^2}{R^2}\right)$.

Find: Total friction force on the fluid from 1 to 2.

Solution:

- First draw a C.V. Again cut through the inlet and outlet 1 and 2. But now, we want to cut along the walls since $F_{friction}$ is our unknown.

- Draw all the forces acting on the C.V. in the x-direction.

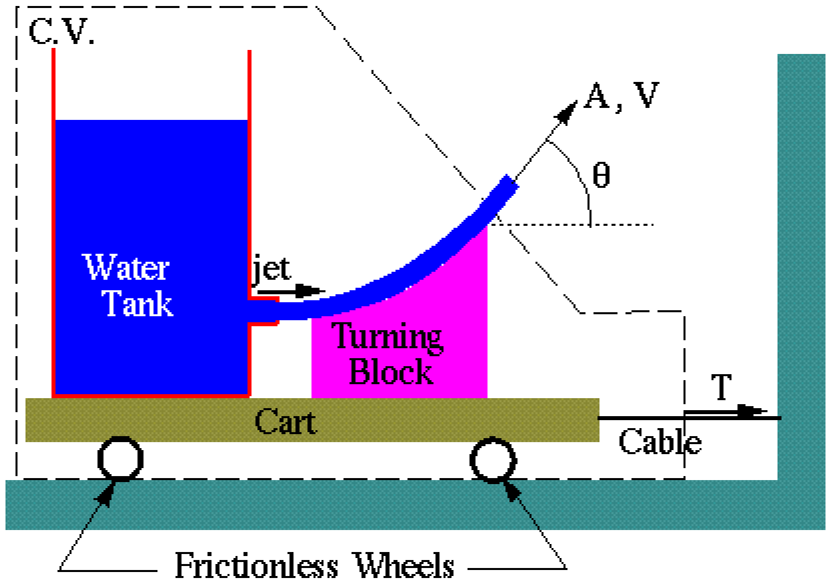


- Simplifications: Assume steady, incompressible, uniform one-dimensional inlet, but not a uniform one-dimensional outlet.
- So, let's use the modified momentum equation for non-1-D inlets/outlets in the x-direction:
  $\sum F_x = \sum F_{x_{grav}} + \sum F_{x_{visc}} + \sum F_{x_{press}} + \sum F_{x_{other}} = \sum_{out} \dot{m} u_{av}\beta - \sum_{in} \dot{m} u_{av}\beta$
- Now look at each term:
  $\sum F_{x_{grav}} = 0$ (no grav. in x-dir)
  $\sum F_{x_{visc}} = -\sum F$ (this is our unknown)
  $\sum F_{x_{press}} = p_1 A - p_2 A$
  $\sum F_{x_{other}} = 0$ (no struts, bolts, etc.)
  $\sum_{in} \dot{m} u_{av}\beta = -\rho U_o^2 A$ (inlet is 1-D so $\beta = 1$)
  $\sum_{out} \dot{m} u_{av}\beta$: This term is the hardest. First of all, what is u_{av}? I.e. what is the equivalent 1-D velocity? Well, since the area at the outlet is the same as that at the inlet, u_{av} at the outlet has to equal U_0.
- What is β at the outlet? By definition, $\beta_2 = \frac{1}{A}\iint_A \left(\frac{u}{u_{av}}\right)^2 dA$. You can plug the equation in and integrate [Try it on your own]. You will get $\beta = \frac{4}{3}$ (the value for laminar pipe flow).
- So, the outlet momentum flux term becomes
  $\sum_{out} \dot{m} u_{av}\beta = \rho U_o^2 A \beta_2$
- Finally, then sum everything and solve for the unknown:
  $\sum F_{friction} = \pi R^2 \; p_1 - p_2 + (1 + \beta_2)\rho U_o^2$
  $\sum F_{friction} = \pi R^2 \left[p_1 - p_2 - \frac{1}{3}\rho U_o^2 \right]$
- Notice, if β_2 is ignored, i.e. (set $\beta_2 = 1$), this last term would cancel completely, and the answer would be wrong.

More Example Problems

Problem # 3.58 in the text:

Given: Cart with water jet, deflector, as shown



Known in this problem are the jet area A, the average velocity V_{av}, the jet deflection angle, θ, and the momentum. flux correction factor of the jet, β_{jet}. Also, frictionless wheels are assumed.

Find: Tension in cable at time t=0.

Solution:

- As always, the first step in any control volume problem is to pick and draw a control volume. The forces and coordinate system have been labeled on the sketch. Here it is appropriate to slice through the jet and slice through the unknown force T (tension in cable), as shown in the sketch.
- Simplifications:
  Incompressible? Yes (water is the fluid, which is approximately an incompressible liquid)
  Steady? - No, not really, since the water level in the tank is falling. But all we are asked for is T at time t=0, so think of this as a pseudo-steady problem in order to find T. Note: The falling water level will have no effect on the x-momentum equation anyway, since its velocity is vertical.

- Now, use the x-component of the momentum equation in it steady form:

$$\sum F_x = \sum F_{x_{grav}} + \sum F_{x_{pres}} + \sum F_{x_{visc}} + \sum F_{x_{other}} = \sum_{outlets} \dot{m} u_{av} \beta - \sum_{inlets} \dot{m} u_{av} \beta$$

no gravity in x-direction; $p = p_a$ everywhere; none in our C.V.; tension, T; no u component at the inlet

There is only one outlet, at which

$$\dot{m} = \rho V_{av} A .$$

The quantity u_{av} in the momentum flux outlet term in the above equation needs to be considered carefully. This is not V_{av}, but rather is the x-component of V_{av}. From a little trig one can see that

$$u_{av} = V_{av} \cos\theta$$

Thus the outlet term on the right hand side of the momentum equation is

$$\rho V_{av}^2 A (\cos\theta) \beta_{jet}$$

and the final form of the x-momentum equation is (solving for T):

$$T = \rho V_{av}^2 A (\cos\theta) \beta_{jet}$$

- Plug in the numbers last: The density of water at room temperature is 998. kg/m^3, the jet velocity, V_{av} is 8 m/s, the cross-sectional area A is that of a circle of diameter 0.04 m, the jet angle is given as 60^o, and the jet momentum flux correction factor is 1.0 for a uniform jet. This yields:

$$T = \left(998 \frac{kg}{m^3}\right)\left(8 \frac{m}{s}\right)^2 \frac{\pi}{4}(0.04 m)^2 \cos 60^o \left[\frac{N \cdot s^2}{kg \cdot m}\right]$$

Or T = 40.1 N.

- <u>Question</u>: At what angle, is the tension a maximum?
<u>Answer</u>: When the jet deflection angle is zero, i.e. the jet simply exits horizontally into the atmosphere. This is clearly seen in the above equation since the cosine of zero is unity.

Example A water-mounted fire pump (an old exam problem.)

<u>Given:</u> A pump is anchored to the ground as shown, with V_j=35.0 m/s and d_j=3.00 cm. Assume the jet has a fully developed turbulent pipe flow profile at its exit.
picture wat_pump.gif

<u>Find:</u> Horizontal force required to hold platform in place.

<u>Solution:</u>

- The first step in any control volume problem is to pick and draw a control volume. As shown, the control volume should slice through jet exit and through the bolts or whatever is holding the platform solidly to the ground. The inlet to the control volume is most easily taken at the surface of the water, where the pressure and velocity are known ($p = p_a$ and V = 0 at the surface). Note: When this problem was given as an exam question, lots of students took their control volume inlet at the pipe inlet. This makes the problem more difficult. Remember the first rule about selecting a control volume - Be Lazy!
- Apply the x-momentum equation:

  $$\sum \mathbf{F}_{x_{grav}} + \sum \mathbf{F}_{x_{press}} + \sum \mathbf{F}_{x_{visc}} + \sum \mathbf{F}_{x_{other}} = \sum_{out} \dot{m}\mathbf{u}_{av}\beta - \sum_{in} \dot{m}\mathbf{u}_{av}\beta$$

  The gravity term on the left hand side is zero because gravity does not act in the x direction. Likewise, the pressure term is zero because everywhere on the control surface above the water, the pressure is atmospheric (including the portion of the control surface that slices through the jet). Below the water, the pressure is hydrostatic, and whatever pressure force is exerted on the left side of the control surface is exactly balanced by that on the right side of the control surface. In the control volume selected, there is no net viscous force acting on the control volume. Thus the first three terms on the left hand side of the x-momentum equation are zero. The only "other" force acting on the control surface is the force of the ground acting on the platform, as shown in the sketch. The direction of this force is assumed to act to the right; if this is wrong, the result will be negative.
  On the right hand side (the momentum flux terms), there is no x-component of velocity at any inlet, so the last term on the right is zero. There is only one outlet, with u_{av} equal to the x-component of the velocity vector of the jet. Thus, the x-momentum equation reduces to

  $$F_{x_{ground\ on\ platform}} = \rho V_{j_{av}}^2 A_j \beta_j \cos\theta$$

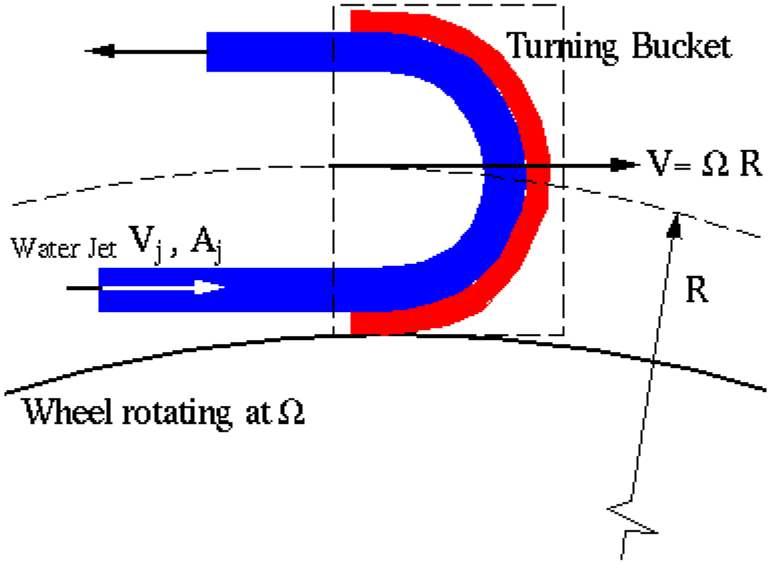
  This is the answer in variable form.
- Lastly, plug in the numbers, noting that for a fully developed turbulent pipe flow the momentum flux correction factor is around 1.02:

  $$F_x = \left(998 \frac{\text{kg}}{\text{m}^3}\right)\left(35.0 \frac{\text{m}}{\text{s}}\right)^2 \left(\frac{\pi}{4}(0.030\text{m})^2\right)(1.02)\left(\cos 25°\right)\left(\frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}\right)$$

  or finally, the horizontal force required to hold platform in place is 799. N.

Problem # 3.51 in the text:

Given: A turbine wheel, powered by a water jet, as shown in the sketch (at time t = 0)



The turbine is spinning at a constant rotational speed.

Assumptions:

- 1-D inlet and outlet
- neglect friction in the turning bucket

(a) Find: The force of the turning bucket on the turbine wheel at this instant of time.

Solution:

- Pick a control volume. Here we will pick a moving control volume, which makes things a little tricky. When the control volume is moving, one must use therelative velocity, i.e. relative to the moving C.V.
- At the inlet, then, the relative velocity at the inlet is obtained by subtracting the velocity of the C.V. from the absolute velocity at the inlet, i.e.
  $\vec{V}_{relative_{inlet}} = V_j \vec{i} - \Omega R \vec{i}$
  where Vj is the magnitude of the absolute velocity at the inlet, and its direction is in the positive x-direction. The second term on the right represents the velocity of the control volume in an absolute reference frame, which is subtracted as shown. The x-component of this relative inlet velocity is simply the magnitude of the above velocity vector, since the vector acts only in the positive x-direction, i.e.
  $u_{relative_{inlet}} = V_j - \Omega R$
- Now apply conservation of mass to this moving control volume. Note: Here, *relative* velocities must be used, since they represent the velocities

actually entering and leaving the control volume as it moves along.

$$\sum_{out} \dot{m} = \sum_{in} \dot{m}, \text{ where } \dot{m} = \rho V_{relative_{inlet}} A$$

$$\rho V_{relative_{outlet}} A = \rho V_{relative_{inlet}} A$$

$$V_{relative_{outlet}} = V_{relative_{inlet}}$$

Note that the areas on either side of the equation are identical by the original assumption of negligible viscous effects. Also the incompressible assumption causes the density to drop out. At time t = 0, when the turning bucket is on the top of the turbine wheel as shown, the velocity of the inlet is in the positive x-direction, while that of the outlet is in the negative x-direction. Thus, the x-component of the relative outlet velocity is simply the negative of that of the relative inlet velocity, i.e.

$$u_{relative_{outlet}} = -(V_j - \Omega R)$$

In other words, the turning bucket changes the direction of the water jet by 180 degrees, but it does not change the magnitude of the jet velocity. (If friction along the walls of the turning vane was taken into account, the jet velocity magnitude would change as well.)

- Now apply conservation of x-momentum for our moving C.V.

$$\sum \mathbf{F}_{x_{grav}} + \sum \mathbf{F}_{x_{press}} + \sum \mathbf{F}_{x_{visc}} + \sum \mathbf{F}_{x_{other}} = \sum_{out} \dot{m}\mathbf{u}_{relative}\,\beta - \sum_{in} \dot{m}\mathbf{u}_{relative}\,\beta$$

Notice that the *relative* x-velocity components are used in the momentum flux terms on the right hand side since this is a moving control volume. The gravity term on the left hand side is zero because gravity does not act in the x direction. Likewise, the pressure term is zero because everywhere on the control surface, the pressure is atmospheric (including the portions of the control surface that slice through the jets). In the control volume selected, there is no net viscous force acting on the control volume. (This would be true even if viscous forces along the turning vane were not ignored, because the control surface does not pass along the turning vane wall.) Thus the first three terms on the left hand side of the x-momentum equation are zero. The only "other" force acting on the control surface is the force of the turbine wheel acting on the turning bucket, which is equal and opposite to the force of the turning bucket acting on the turbine wheel, as shown in the sketch. The direction of the force of the bucket on the wheel is assumed to act to the right; if this is wrong, the result will be negative.

On the right hand side (the momentum flux terms), there is only one inlet and one outlet. At both the inlet and the outlet, the x-component of the relative velocity vector of the jet is known from the conservation of mass analysis above. Since friction is being neglected, it is assumed that the momentum flux correction factors are unity. Thus, the x-momentum equation reduces to

$$F_{\text{bucket on wheel}} = \dot{m}\left(u_{\text{relative}_{\text{inlet}}} - u_{\text{relative}_{\text{outlet}}}\right)$$

$$F_{\text{bucket on wheel}} = \rho\left(V_j - \Omega R\right)A_j\left[\left(V_j - \Omega R\right) - (-)\left(V_j - \Omega R\right)\right]$$

$$F_{\text{bucket on wheel}} = 2\rho A_j\left(V_j - \Omega R\right)^2$$

This is the force of the turning bucket on the turbine wheel.

(b) Find: The power, P, delivered to the wheel at this instant of time (t = 0).

Solution:

- Power on a rotating wheel is defined as the torque times the angular velocity of the wheel. The torque due to this turning bucket is the force just derived times the radius fo the wheel, R. Thus,

$$P = 2\rho A_j\left(V_j - \Omega R\right)^2 R\Omega$$

(c) Find: The angular velocity which provides the maximum power to the wheel.

Solution:

- To find the maximum power, take the derivative of power with respect to angular velocity.

$$\frac{dP}{d\Omega} = 2\rho A_j R\frac{d}{d\Omega}\left[\left(V_j - \Omega R\right)^2\Omega\right]$$

$$= 2\rho A_j R\left[2\Omega\left(V_j - \Omega R\right)(-R) + \left(V_j - \Omega R\right)^2\right]$$

$$= 2\rho A_j R\left(V_j - \Omega R\right)\left[-2\Omega R + \left(V_j - \Omega R\right)\right]$$

- The values of angular velocity at which this derivative is zero then represent conditions where the power is either a minimum or a maximum. There are two such roots, i.e.

$$\Omega = \frac{V_j}{R} \quad \left(V_j = \Omega R\right) \quad \text{or} \quad \Omega = \frac{V_j}{3R} \quad \left(V_j = 3\Omega R\right)$$

- The first of these represents the case where the absolute jet speed and the speed of the turning bucket are the same. Under such conditions, no power is transferred to the turbine wheel at all. (This can be verified easily, i.e. P = 0 when the first root is plugged into the result of part (b) above.) In other words, the first root is a minimum. The second root is therefore the desired one; i.e. the angular velocity which supplies the maximum power to the turbine wheel is that in which the turbine wheel spins three times slower at its rim than the jet speed, i.e.

$$\Omega = \frac{V_j}{3R}$$

Source:

http://www.mne.psu.edu/cimbala/Learning/Fluid/CV_Momentum/index.htm