# Conservation of Mass using Control Volumes

## Conservation of Mass

In the Reynolds transport theorem for conservation of mass, let B = m, i.e. $B_{sys} = m_{sys}$ = mass of the system,

$$\beta = \frac{m_{sys}}{m} = 1$$

For our system, we know that

$$\frac{dm_{sys}}{dt} = 0$$

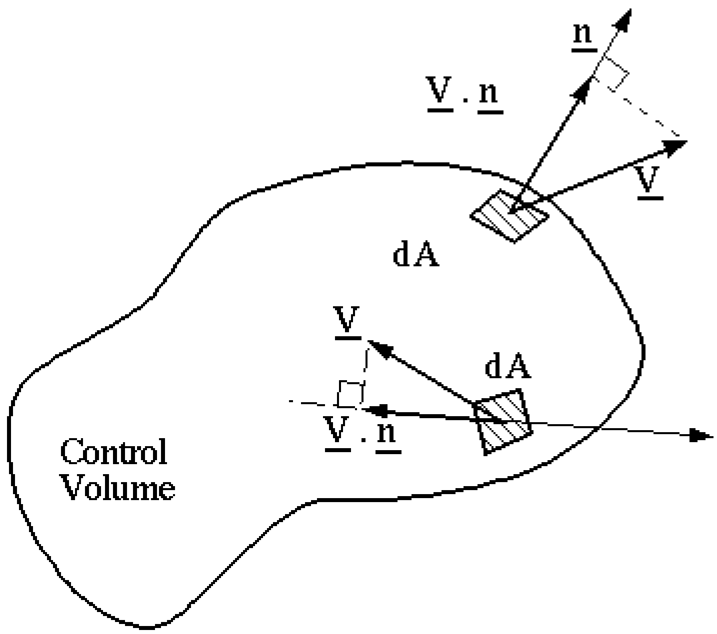
So, Reynold's Transport Theorem (R.T.T.) becomes

$$\frac{dm_{sys}}{dt} = 0 = \frac{d}{dt}\int_{cv} \rho \cdot dv + \oint_{cs} \rho \cdot \mathbf{V} \bullet \mathbf{n} dA$$

OR,

$$0 = \int_{cv} \frac{\partial \rho}{dt} dV + \oint_{cs} \rho \mathbf{V} \bullet \mathbf{n} dA$$

- Let's analyze the last term in the equation:
    - n = outward normal unit vector (perpendicular to surface area element dA)
    - **V** = velocity vector, which can be in any arbitrary direction




- Recall, from definition of dot product:
  - Outlets: **V** is <u>outward</u> (mass leaving C.V.) $\mathbf{V} \bullet \mathbf{n} > 0$
  - Inlets: **V** is <u>inward</u> (mass entering C.V.) $\mathbf{V} \bullet \mathbf{n} < 0$
- Now, it turns out that

$$\dot{m} = \int_{surface} \rho \cdot \mathbf{V} \bullet \mathbf{n} dA$$ = mass flow rate outward through a surface

so, if we integrate over the entire surface,

$$\oint_{CS} \rho \cdot \mathbf{V} \bullet \mathbf{n} dA = \sum_{outlets} \dot{m} - \sum_{inlets} \dot{m}$$

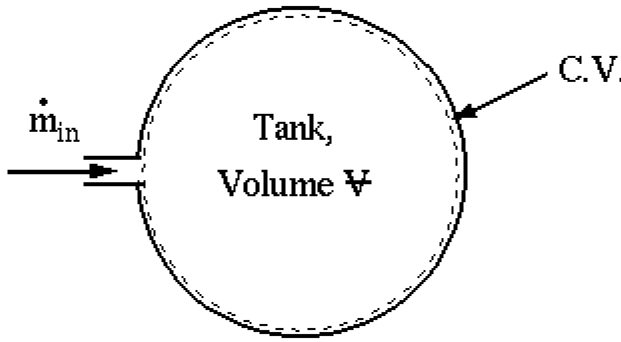
and the conservation of mass equation becomes:

$$0 = \frac{d}{dt}\int_{CV} \rho \cdot dV + \sum_{outlets} \dot{m} - \sum_{inlets} \dot{m}$$

*this is the most useful form.

- e.g. <u>Given</u>: A rigid tank of volume V with p = $p_0$ at t = 0. Air is pumped in at constant mass flow rate isothermally.

Find: p in tank as a function of time
Solution: first, draw a C.V. inside the entire tank.




Now use our conservation of mass equation.

$$0 = \frac{d}{dt}\int_{CV} \rho \cdot dV + \sum_{outlets} \dot{m} - \sum_{inlets} \dot{m}$$

assume density is equal everywhere in the tank, and only varies with time. There are no outlets and the only term remaining is the mass term at the inlet:

$$0 = \frac{d}{dt}\int_{CV} \rho \cdot dV - \dot{m}_{in}$$

This is a differential equation we must solve by separating the variables and integrating from t=0 to some arbitrary t.

$$\rho = \rho_0 + \frac{\dot{m}_{in}}{Volume} \cdot t$$

Finally, use the ideal gas law to get the pressure. Thus,

$$p = p_0 + \frac{\dot{m}_{in}}{Vol.} \cdot tRT$$

(NOTE: pressure varies linearly with t)

**Special Case**: **STEADY FLOW**

For a steady flow, nothing is a function of time, the d/dt term in the conservation of mass equation goes away.

$$\sum_{outlets} \dot{m} = \sum_{inlets} \dot{m}$$

in other words, "What goes in must come out"

If more is going in than out, mass will be accumulating with time inside the control volume, and it would not be steady state!! ( and you need to include the unsteady volume term)

**Incompressible Steady Flow** ( density = constant)

- Since the density is constant,

$$\dot{m} = \int_{surface} \rho \cdot \mathbf{V} \bullet \mathbf{n} dA$$

take density outside of the integral

$$\dot{m} = \rho \int_{surface} \mathbf{V} \bullet \mathbf{n} dA$$

So the integral term is equal to Q, the volume flow rate

i.e. $\dot{m} = \rho Q$ where mass flow rate = density X volume flow rate

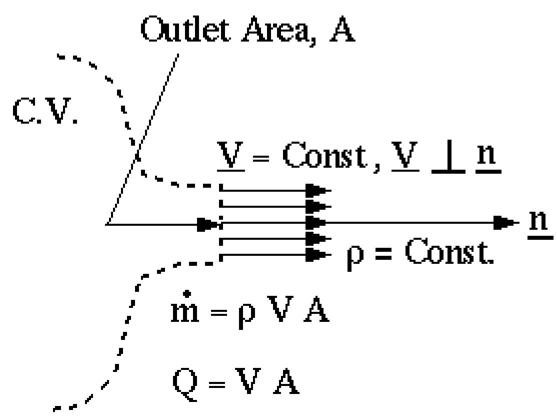
So the conservation of mass equation becomes:

$$\sum_{outlets} Q = \sum_{inlets} Q$$

"volume flow rate it" = "volume flow rate out"

- **Another simplification: 1-D inlets & outlets** (for incompressible flow)

  Consider an outlet:




Where:

- 1-D outlet means velocity is parallel to **n**, the unit normal vector.
- **V** = constant across the outlet.
- Density = constant across the outlet

thus, $Q_{out}$ = **V** A. Therefore, at an <u>inlet</u>

$$\mathbf{V} \bullet \mathbf{n} = -\mathrm{V} \quad \text{and} \quad \dot{m} = -\rho \mathrm{V} A$$

we usually say:

$$\dot{m}_{in} = \rho \mathrm{V} A$$
$$Q_{in} = \mathrm{V} A$$

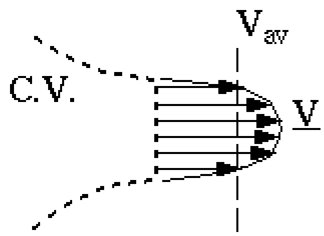
….since the negative sign is accounted for in the conservation of mass equation,

i.e. $\sum_{outlets} \dot{m} - \sum_{inlets} \dot{m} = 0$

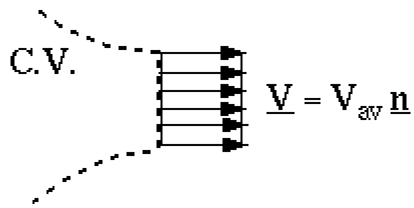
- **<u>Average Velocity</u>**

If an inlet or exit is <u>not</u> 1-D, we can still use $\dot{m} = \rho \mathrm{V} A$ but $\mathbf{V}_{AV}$ must be the <u>average velocity</u>

- e.g.: At an outlet, for incompressible flow ( density = constant)




Actual Outlet ( Non – Uniform)




Equivalent 1–D Outlet

Let us define:

$$V_{AV} = \frac{1}{A}\int \mathrm{V} \cdot n \cdot dA$$

*OR*

$$V_{AV} = Q/A$$

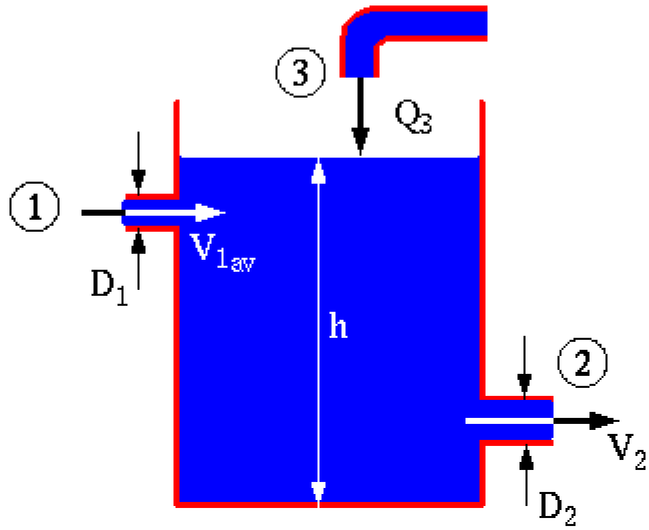
but for most problems **V** is parallel to **n**……

$$V_{AV} = \frac{1}{A}\int \mathrm{V} \cdot dA$$

An equivalent 1-D outlet will have the same mass flow rate as the actual outlet.

- **<u>EXAMPLE</u>** 3.14 in text:

  A container of water,




Given: $\mathbf{V}_{AV}$, $Q_3$, $D_1$, $D_2$ and h equal constants.
$V_1 = 3$ m/s; $Q_3 = 0.01\ m^3/s$; h = constant; $D_1 = 0.05m$; $D_2 = 0.07m$

Find: Average exit velocity $V_2$

Solution: Use conservation of mass. First draw the C.V. shown.

Since it is steady and incompressible, $\sum_{in} Q = \sum_{out} Q$

$$\sum_{out} Q = \mathbf{V}_2 \cdot A_2 = \mathbf{V}_2 \pi \frac{D_2^2}{4}$$

$$\sum_{in} Q = \mathbf{V}_2 \cdot A_2 = \mathbf{V}_2 \pi \frac{D_2^2}{4} + Q_3$$

Using the above equations, solve for $V_2$:

$$\mathbf{V}_2 = \frac{\mathbf{V}_1 \pi \frac{D_1^2}{4} + Q_3}{\pi \frac{D_2^2}{4} + Q_3} = 4.13 \,{}^{m}\!/_{s}$$

Source:

http://www.mne.psu.edu/cimbala/Learning/Fluid/CV_Mass/index.htm