Basics of Vibrations for Simple Mechanical Systems

Introduction:

The fundamentals of Sound and Vibrations are part of the broader field of mechanics, with strong connections to classical mechanics, solid mechanics and fluid dynamics. Dynamics is the branch of physics concerned with the motion of bodies under the action of forces. Vibrations or oscillations can be regarded as a subset of dynamics in which a system subjected to restoring forces swings back and forth about an equilibrium position, where a system is defined as an assemblage of parts acting together as a whole. The restoring forces are due to elasticity, or due to gravity.

The subject of Sound and Vibrations encompasses the generation of sound and vibrations, the distribution and damping of vibrations, how sound propagates in a free field, and how it interacts with a closed space, as well as its effect on man and measurement equipment. Technical applications span an even wider field, from applied mathematics and mechanics, to electrical instrumentation and analog and digital signal processing theory, to machinery and building design. Most human activities involve vibration in one form or other. For example, we hear because our eardrums vibrate and see because light waves undergo vibration. Breathing is associated with the vibration of lungs and walking involves (periodic) oscillatory motion of legs and hands. Human speak due to the oscillatory motion of larynges (tongue).

In most of the engineering applications, vibration is signifying to and fro motion, which is undesirable. Galileo discovered the relationship between the length of a pendulum and its frequency and observed the resonance of two bodies that were connected by some energy transfer medium and tuned to the same natural frequency. Vibration may results in the failure of machines or their critical components. The effect of vibration depends on the magnitude in terms of displacement, velocity or accelerations, exciting frequency and the total duration of the vibration. In this chapter, the vibration of a single-degree-of-freedom (SDOF), Two degree of freedom system with and without damping and introductory multi-degree of freedom system will be discussed in this section.

1. LINEAR SYSTEMS

Often in Vibrations and Acoustics, the calculation of the effect of a certain physical quantity termed as the input signal on another physical quantity, called the output signal; (Figure 1-1). An example is that of calculating vibration velocity *v(t)*, which is obtained in a structure when it is excited by a given force *F(t)*. That problem can be solved by making use of the theory of *linear time- invariant systems*.

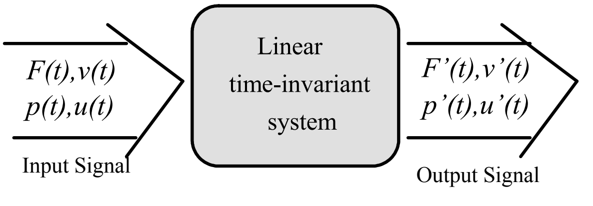
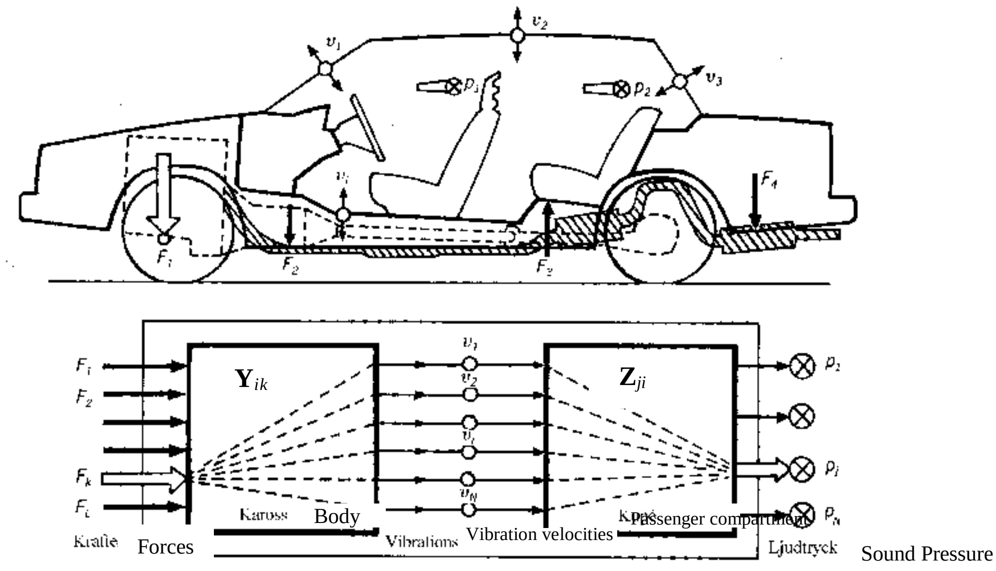


Fig. 0-1 A linear time-invariant system describes the relationship between an input signal and an output signal. For example, the input signal could be a velocity v(t), and the output signal a force F(t), or the input signal an acoustic pressure p(t) and the output signal an acoustic particle velocity u'(t). [Sound and vibration book by KTH[1]]

From a purely mathematical standpoint, a linear system is defined as one in which the relationship between the input and output signals can be described by a linear differential equation. If the coefficients are, moreover, independent of time, i.e., constant, then the system is also time invariant. A linear system has several important features.

Example 0-1 [1]

The figure below, from the introduction, shows an example in which the forces that excite an automobile are inputs to a number of linear systems, the outputs from which are vibration velocities at various points in the structure. The vibration velocities are then, in turn, inputs to a number of linear systems, the outputs from which are sound pressures at various points in the passenger compartment. By adding up the contributions from all of the significant excitation forces, the total sound pressures at points of interest in the passenger compartment can be found. The engine is fixed to the chassis via vibration isolators. If the force F_1 that influences the chassis can be cut in half, then, for a linear system, all vibration velocities $v_1 - v_N$ caused by the force F_1 are also halved. In turn, the sound pressures $p_1 - p_N$, which are brought about by the velocities $v_1 - v_N$, are halved as well. In this chapter, *linear oscillations* in mechanical systems are considered, i.e., oscillations in systems for which there is a linear relation between an exciting force and the resulting motion, as described by displacements, velocities, and accelerations. Linearity is normally applicable whenever the kinematic quantities can be regarded as small variations about an average value, implying that the relation between the input signal and the output signal can be described by linear differential equations with constant coefficients.



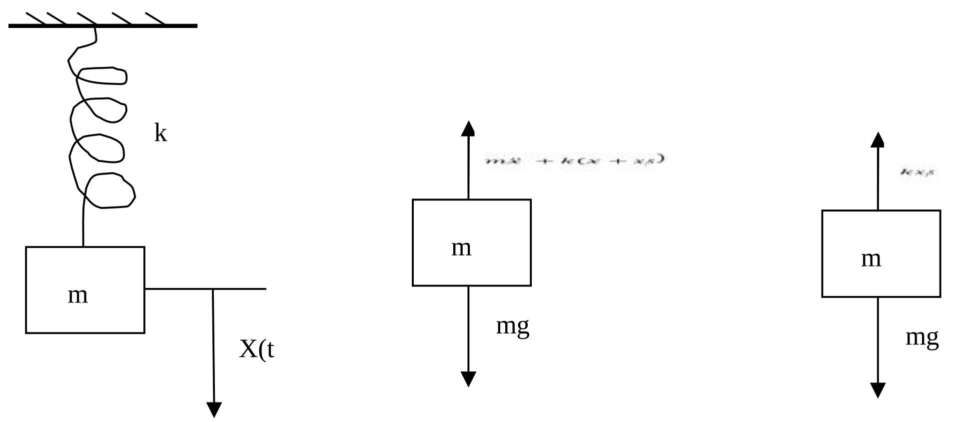
(Picture: Volvo Technology Report, nr 1 1988) [1]

1.1 SINGLE DEGREE OF FREEDOM SYSTEMS

In basic mechanics, one studies *single degree-of-freedom systems* thoroughly. One might wonder why so much attention should be given to such a simple problem. The single degree-of-freedom system is so interesting to study because it gives us information on how a system's characteristics are influenced by different quantities. Moreover, one can model more complex systems, provided that they have isolated resonances, as sums of simple single degree-of-freedom systems.

1.2 **Spring Mass System**

Most of the system exhibit simple harmonic motion or oscillation. These systems are said to have elastic restoring forces. Such systems can be modeled, in some situations, by a spring-mass schematic, as illustrated in Figure 1.2. This constitutes the most basic vibration model of a machine structure and can be used successfully to describe a surprising number of devices, machines, and structures. This system provides a simple mathematical model that seems to be more sophisticated than the problem requires. This system is very useful to conceptualize the vibration problem in different machine components.



(a) (b) (c)

Fig.1.2 (a) Spring-mass schematic (b) free body diagram, (c) free body diagram in static condition

If $x = x(t)$ denotes the displacement (m) of the mass m (kg) from its equilibrium position as a function of time t (s), the equation of motion for this system becomes,

$$m\ddot{x} + k(x + x_s) - mg = 0 \qquad (1.1)$$

where k =the stiffness of the spring (N/m),

x_s = static deflection

m = the spring under gravity load,

g = the acceleration due to gravity (m/s2),

$\ddot{x}$ = acceleration of the system

Applying static condition as shown in Fig. 1.2 (c) the equation of motion of the system yields

$$m\ddot{x} + kx = 0 \qquad (1.2)$$

This equation of motion of a single-degree-of-freedom system and is a linear, second-order, ordinary differential equation with constant coefficients. A simple experiment for determining the spring stiffness by adding known amounts of mass to a spring and measuring the resulting static deflection x_s is shown in Fig. 1.3. The results of this static experiment can be plotted as force (mass times acceleration) v/s x_s, the slope yielding the value of spring stiffness k for the linear portion of the plot as illustrated in Figure 1.4.

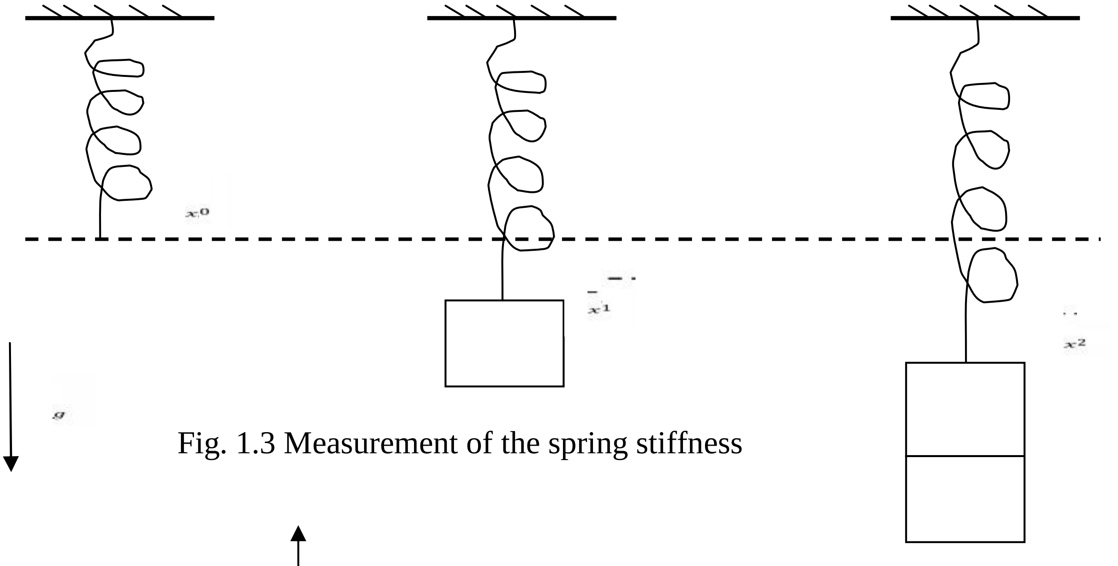


Fig. 1.3 Measurement of the spring stiffness

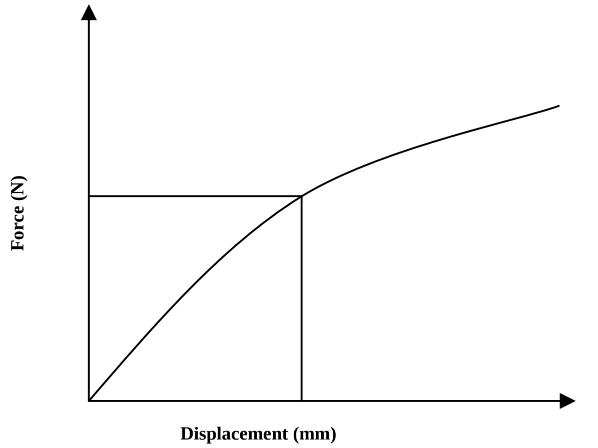


Fig. 1.4 Determination of the spring stiffness

Once m and k are determined from static experiments, Equation (1.2) can be solved to yield the time history of the position of the mass m, given the initial position and velocity of the mass. The form of the solution of previous equation is found from substitution of an assumed periodic motion as,

$$x(t) = A \sin(\omega_n t + \phi) \qquad (1.3)$$

Where, $\omega_n = \sqrt{k/m}$ is the natural frequency (rad/s).

Here, A= the amplitude

Φ= phase shift,

A and Φ are constants of integration determined by the initial conditions.

If x_0 is the specified initial displacement from equilibrium of mass m, and v_0 is its specified initial velocity, simple substitution allows the constants A and Φ to be obtained. The unique displacement may be expressed as,

$$x(t) = \sqrt{\frac{\omega_n^2 x_0^2 + \upsilon_0^2}{\omega_n^2}} \sin[\omega_n t + \tan^{-1}\left(\frac{\omega_n x_0}{\upsilon_0}\right)] \qquad (1.4)$$

Or,

$$x(t) = \frac{\upsilon_0}{\omega_n} \sin \omega_n t + x_0 \cos \omega_n t$$

Equation 1.2 can also be solved using a pure mathematical approach as described follows.

Substituting $x(t) = C\, e^{\lambda t}$

$$m\lambda^2 e^{\lambda t} + k e^{\lambda t} = 0 \qquad (1.5)$$

Here $C \neq 0$ and $e^{\lambda t} \neq 0$, Hence $m\lambda^2 + k = 0$

Or

$$\lambda = \pm j \left(\frac{k}{m}\right)^{1/2} = \pm \omega_n j$$

where, j is an imaginary number $= \sqrt{-1}$

Hence the generalized solution yields as,

$$x(t) = C_1 e^{\omega_n j t} + C_2 e^{-\omega_n j t} \qquad (1.6)$$

where C_1 and C_2 are arbitrary complex conjugate constants of integration.

The value of the constants C_1 and C_2 can be determined by applying the initial conditions of the system. Note that the equation 1.2 is valid only as long as spring is linear.

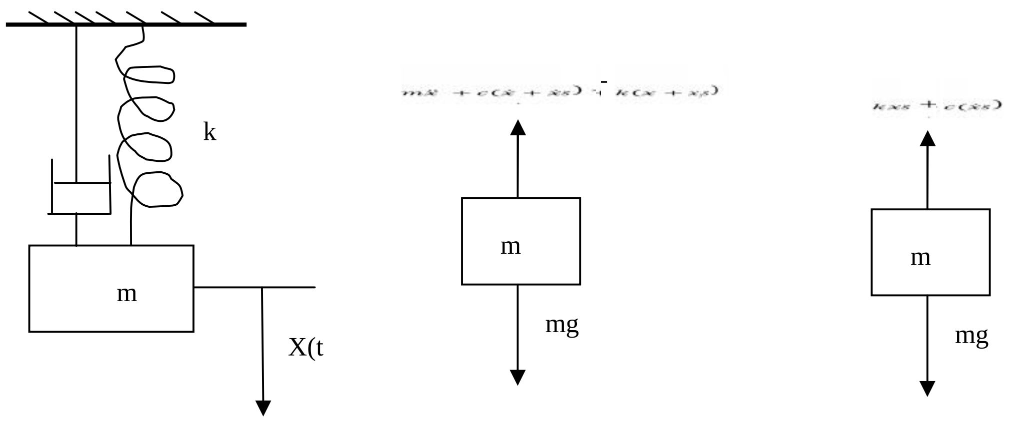
1.3 Spring Mass Damper system

Most systems will not oscillate indefinitely when disturbed, as indicated by the solution in Equation (1.4). Typically, the periodic motion damped out after some time. The easiest way to model this mathematically is to introduce a new term, named as damping force term, into Equation (1.2).

Incorporating the damping term in equation (1.2) yield as,

$$m\ddot{x} + c\dot{x} + kx = 0 \qquad (1.7)$$

Physically, the addition of a dashpot or damper results in the dissipation of energy, as illustrated in Figure 1.5



(a) **(b)** **(c)**

Fig. 1.5 (a) Schematic of the spring–mass–damper system, (b) free body diagram of the system in part (a), (c) free body diagram due to static condition

If the dashpot exerts a dissipative force proportional to velocity on the mass m, the equation (1.7) describes the equation of the motion. Unfortunately, the constant of proportionality, c, cannot be measured by static methods as m and k are measured in spring mass system.

The constant of proportionality c is known as damping coefficient and its unit in MKS is Ns/m. A general mathematical approach can be used to solve the equation 1.7 as described below.

Substituting, $x(t) = a\, e^{\lambda t}$ in equation 1.7, get,

$$a(m\lambda^2 e^{\lambda t} + c\lambda e^{\lambda t} + k e^{\lambda t}) = 0 \qquad (1.8)$$

here $a \neq 0$ and $e^{\lambda t} \neq 0$

hence, $m\lambda^2 + c\lambda + k = 0$

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0 \quad (1.9)$$

The solution of equation 1.8 yields as follows

$$\lambda_{1,2} = -\frac{c}{2m} \pm \frac{1}{2}\sqrt{\frac{c^2}{m^2} - 4\frac{k}{m}}$$

The quantity under the radical is called the discriminant. The value of the discriminant decides that whether the roots are real or complex. Damping ratio: It is relatively convenient to define a non-dimensional quantity named as damping ratio. The damping ratio is generally given by symbol Zeeta (ξ) and mathematically defined as;

$$\xi = \frac{c}{2\sqrt{km}}$$

Substituting the value of k ,m and c in terms of ξ and ω_n, the equation (1.7) yields as,

$$\ddot{x} + 2\xi\,\omega_n\dot{x} + \omega_n^2 x = 0 \qquad (1.10)$$

And equation (1.9) yields as

$$\lambda_{1,2} = -\xi\,\omega_n \pm \omega_n\sqrt{\xi^2 - 1} = -\xi\,\omega_n \pm \omega\, j \qquad (1.11)$$

where, ω is the damped natural frequency for ($0 < \xi < 1$) the damped natural frequency is defined as $\omega = \omega_n\sqrt{1 - \xi^2}$

Clearly, the value of the damping ratio,(ξ), determines the nature of the solution of Equation (1.6).

“Fundamentals of Sound and Vibrations” by KTH Sweden [1], this book is used under IITR-KTH MOU for course development.

Source:
http://nptel.ac.in/courses/112107088/1