Input/output and state space models in Dynamics Non Linear Systems

This lecture presents some basic definitions and simple examples on nonlinear dynamical systems modeling.

1.1 Behavioral Models.

The most general (though rarely the most convenient) way to define a system is by using a behavioral input/output model.

1.1.1 What is a signal?

In these lectures, a *signal* is a *locally integrable* function $z:\ \mathbf{R}_+ \mapsto \mathbf{R}^k$, where $\mathbf{R}_+$ denotes the set of all non-negative real numbers. The notion of "local integrability" comes from the Lebesque measure theory, and means simply that the function can be safely and meaningfully integrated over finite intervals. Generalized functions, such as the delta function $\delta(t)$, are not allowed. The argument $t \in \mathbf{R}_+$ of a signal function will be referred to as "time" (which it usually is).

Example 1.1 Function $z = z(\cdot)$ defined by

$$z(t) = \begin{cases} t^{-0.9}\mathrm{sgn}(cos(1/t)) & \text{for } t > 0, \\ 0 & \text{for } t = 0 \end{cases}$$

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is a valid signal, while

$$z(t) = \begin{cases} 1/t & \text{for } t > 0, \\ 0 & \text{for } t = 0 \end{cases}$$

and $z(t) = \dot{\delta}(t)$ are not.

The definition above formally covers the so-called *continuous time* (CT) signals. Discrete time (DT) signals can be represented within this framework as special CT signals. More precisely, a signal $z : \mathbf{R}_+ \mapsto \mathbf{R}^k$ is called a *DT signal* if it is constant on every interval $[k, k+1)$ where $k = 0, 1, 2, \ldots$.

1.1.2 What is a system?

Systems are objects producing signals (called *output signals*), usually depending on other signals (*inputs*) and some other parameters (*initial conditions*). In most applications, mathematical models of systems are defined (usually implicitly) by *behavior sets*. For an *autonomous* system (i.e. for a system with no inputs), a behavior set is just a set $\mathcal{B} = \{z\}$ consisting of some signals $z : \mathbf{R}_+ \mapsto \mathbf{R}^k$ (k must be the same for all signals from $\mathcal{B}$). For a system with input v and output w, the behavior set consists of all possible input/output pairs $z = (v(\cdot), w(\cdot))$. There is no real difference between the two definitions, since the pair of signals $z = (v(\cdot), w(\cdot))$ can be interpreted as a single vector signal $z(t) = [v(t); w(t)]$ containing both input and output stacked one over the other.

Note that in this definition a fixed input $v(\cdot)$ may occur in many or in no pairs $(v, w) \in \mathcal{B}$, which means that the behavior set does not necessarily define system output as a function of an arbitrary system input. Typically, in addition to knowing the input, one has to have some other information (initial conditions and/or uncertain parameters) to determine the output in a unique way.

Example 1.2 The familiar *ideal integrator* system (the one with the transfer function $G(s) = 1/s$) can be defined by its behavioral set of all input/output scalar signal pairs (v, w) satisfying

$$w(t_2) - w(t_1) = \int_{t_1}^{t_2} v(\tau)d\tau, \quad \forall\ t_1, t_2 \in [0, \infty).$$

In this example, to determine the output uniquely it is sufficient to know v and $w(0)$.

In Example 1.1.2 a system is characterised by an integral equation. There is a variety of *other* ways to define the *same* system (by specifying a transfer function, by writing a differential equation, etc.)

1.1.3 What is a linear/nonlinear system?

A system is called *linear* if its behavior set satisfies linear superposition laws, i.e. when for every $z_1, z_2 \in \mathcal{B}$ and $c \in \mathbf{R}$ we have $z_1 + z_2 \in \mathcal{B}$ and $cz_1 \in \mathcal{B}$.

Excluding some absurd examples[2], linear systems are those defined by equations which are linear with respect to v and w. In particular, the ideal integrator system from Example 1.1.2 is linear.

A *nonlinear system* is simply a system which is not linear.

1.2 System State.

It is important to realize that system state can be defined for an arbitrary behavioral model $B = \{z(\cdot\}$.

1.2.1 Two signals defining same state at time t.

System state at a given time instance t_0 is supposed to contain all information relating past ($t < t_0$) and future ($t > t_0$) behavior. This leads us to the following definitions.

Definition Let $\mathcal{B}$ be a behavior set. Signals $z_1, z_2 \in \mathcal{B}$ are said to *commute* at time t_0 if the signals

$$z_{12}(t) = \begin{cases} z_1(t) & \text{for } t \le t_0, \\ z_2(t) & \text{for } t > t_0 \end{cases}$$

and

$$z_{21}(t) = \begin{cases} z_2(t) & \text{for } t \le t_0, \\ z_1(t) & \text{for } t > t_0 \end{cases}$$

also belong to the behavior set.

Definition Let $\mathcal{B}$ be a behavior set. Signals $z_1, z_2 \in \mathcal{B}$ are said to *define same state of $\mathcal{B}$ at time* t_0 if the set of $z \in \mathcal{B}$ commuting with z_1 at t_0 is the same as the set of $z \in \mathcal{B}$ commuting with z_2 at t_0.

Definition Let $\mathcal{B}$ be a behavior set. Let X be any set. A function $x : \mathbf{R} \times \mathcal{B} \mapsto X$ is called a *state* of system $\mathcal{B}$ if z_1 and z_2 define same state of $\mathcal{B}$ at time t whenever $x(t, z_1(\cdot)) = x(t, z_2(\cdot))$.

Example 1.3 Consider a system in which both input v and output w are *binary* signals, i.e. DT signals taking values from the set $\{0, 1\}$. Define the input/output relation by the following rules: $w(t) = 1$ *only if* $v(t) = 1$, and for every $t_1, t_2 \in \mathbf{Z}_+$ such that

[2] Such as the (linear) system defined by the nonlinear equation $(v(t) - w(t))^2 = 0 \; \forall \, t$

$w(t_1) = w(t_2) = 1$ and $w(t) = 0$ for all $t \in (t_1, t_2) \cap \mathbf{Z}$, there are exactly two integers t in the interval (t_1, t_2) such that $v(t) = 1$.

In other words, the system counts the 1's in the input and, every time the count reaches three, the system resets its counter to zero, and outputs 1 (otherwise producing 0's).

It is easy to see that two input/output pairs $z_1 = (v_1, w_1)$ and $z_2 = (v_2, w_2)$ commute at a (discrete) time t_0 if and only if $N(t_0, z_1) = N(t_0, z_2)$, where $N(t_0, z)$ for $z = (v, w) \in \mathcal{B}$ is the number of 1's in $v(t)$ for $t \in (t_0, t_1) \cap \mathbf{Z}$, where t_1 means the next (after t_0) integer time t when $w(t) = 1$. Hence the state of the system can be defined by a function $x: \mathbf{R}_+ \times \mathcal{B} \mapsto \{0, 1, 2\}$, $x(t, z) = N(t, z)$.

In this example, knowing a system state allows one to write down *state space equations* for the system:

$$x(t+1) = f(x(t), v(t)), \quad w(t) = g(x(t), v(t)), \tag{1.1}$$

where

$$f(x, v) = (x + v)\text{mod}3,$$

and $g(x, v) = 1$ if and only if $x = 2$ and $v = 1$.