Vertical Vibration of Founations on Homogeneous Elastic Half-Space

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Abstract— ***The stiffness and damping coefficients for a rigid massless circular foundation resting on homogeneous elastic half-space under vertical harmonic excitation are evaluated using one-dimensional wave propagation in cones (cone model), based on strength of material approach. Using these coefficients, the frequency dependant displacement amplitude as well as the magnification factor of a massive foundation is computed. The resonant frequencies and peak amplitudes are also studied varying the influencing parameters such as the type of soil and the mass of the foundation. To verify the validity of the model developed, the computed response is compared with reported analytical results with wide variation of influencing parameters, which shows a good agreement. Thus, the simplified cone model can be used in day-to-day analysis and design of foundations under vertical dynamic loading.***

Keywords-c*one model, foundation vibration, half-space, impedance functions, wave propagation*

I. Introduction

The determination of resonant frequency and resonant amplitude of foundations has been a subject of considerable interest in the recent years, in relation to the design of machine foundations, as well as the seismic design of important massive structures such as nuclear power plants. One of the key steps in the current methods of dynamic analysis of a foundation soil system under machine type loading is to estimate the dynamic impedance functions (spring and dashpot coefficients) of an 'associated' rigid but massless foundation. With the help of these functions the amplitude of vibration is calculated using the equation of motion of a single degree of freedom oscillator.

Over the years, a number of methods have been developed for foundation vibration analysis; namely, (i) Elastic half-space theory; (ii) Spring-mass-dashpot model; (iii) Half-space analog model; (iv) Lumped parameter model; (v) Numerical / semi-analytical methods like finite element method, boundary element method, scaled boundary finite element method and thin layer method. Each method has its own advantages and its limitations.

The solution of the "dynamic Boussinesq" problem of Lamb [1] formed the basis for the study of oscillation of footings resting on a half-space (Reissner [2]; Sung [3]; Richart et al. [4]). Reissner [2] developed the first analytical solution for a vertically loaded cylindrical disk on elastic half-space assuming uniform stress distribution under the footing. Later, extending Reissner's solution, many investigators (Bycroft [5], Lysmer and Richart [6], Luco and Westman [7], Nagendra [8], Wolf [9], Luco and Mita [10], Pradhan [11] to name a few) studied different modes of vibrations with different contact stress distributions. Most recently, Chen and Shi [12] developed a simplified model (optimal equivalent model) with frequency independent parameters, which can effectively simulate the unbounded soil for a rigid disk lying on uniform elastic half-space.

Numerical/semi-analytical methods though very accurate are not always warranted because of the complexities involved in the problem, particularly in the soil properties. Therefore a number of simplified approximate methods have been developed along with the exact solutions. Cone model is one of such approximate analytical methods, wherein elastic half-space is truncated into a semi-infinite cone and the principle of one-dimensional wave propagation through this cone (Beam with varying cross-section) is considered. The work reported in this paper studies the dynamic response of foundation resting on elastic homogeneous half-space using the simplified cone model.

II. Problem Formulation

A rigid massless circular foundation of radius r_0 resting on a homogeneous half-space is addressed for vertical dynamic excitation (Fig. 1). The soil is assumed to be elastic with shear modulus *G*, Poisson's ratio *n* and mass density *r*. The interaction force P_0 and the corresponding displacement u_0 are assumed to be harmonic. The dynamic impedance of the massless foundation (disk) is expressed by

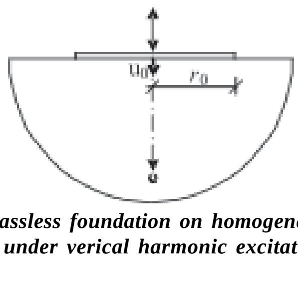

Figure 1. ***Massless foundation on homogeneous half=space under verical harmonic excitation***

$$\overline{K}(a_0) = P_0 / u_0 = K\,[k(a_0) + i a_0 c(a_0)] \qquad (1)$$

where, $\overline{K}(a_0) =$ dynamic impedance, $k(a_0) =$ spring coefficient, $c(a_0) =$ damping coefficient $a_0 = \omega r_0 / c_s$, dimensionless frequency with $c_s = \sqrt{G/\rho}$, shear wave velocity of the soil. $K = 4Gr_0/(1-\upsilon) =$ static stiffness coefficient of disk on homogeneous half-space.

Using the equations of dynamic equilibrium, the dynamic displacement amplitude of a massive foundation with mass m and subjected to a vertical harmonic force Q is expressed as

$$|u_0| = \left|\frac{Q}{K[k(a_0) + i a_0 c(a_0) - B a_0^2]}\right| \qquad (2)$$

Where, $|u_0|$ = dynamic displacement amplitude under the foundation resting on the homogeneous half-space. $|Q|$ = force amplitude and $B = \frac{G r_0}{K} b_0$, with $b_0 = m/\rho r_0^3$, the mass ratio. In general, $|Q|$ can be assumed to be constant or equal to $m_e e \omega^2$ which is generated by the eccentric rotating part in the machine, where m_e is the eccentric mass, e is the eccentricity and ω is the circular frequency.

Dynamic displacement amplitude given in Eq. (2) can be expressed in the non-dimensional form as given below,

$$\left|\frac{u_0 G r_0}{Q}\right| = \frac{G r_0}{K}\left|[k(a_0) + i a_0 c(a_0) - B a_0^2]\right|^{-1} \qquad (3)$$

The magnification factor i.e. the ratio of dynamic displacement to the static displacement is expressed by

$$M = \left|\frac{u_0}{Q/K}\right| = \left|[k(a_0) + i a_0 c(a_0) - B a_0^2]\right|^{-1} \qquad (4)$$

I. Cone Model For Vertical Translation

The theory of wave propagation in a semi-infinite truncated cone is presented based on strength of material approach. This one sided cone is directly used to model a disk on the surface of a homogeneous half-space.

A. Equation of Motion

The translational truncated semi-infinite cone with the apex height z_0 and radius r_0 is shown for axial distortion in Fig. 2, which is used to model the vertical degree of freedom of a circular disk of radius r_0 on the surface of a half-space.

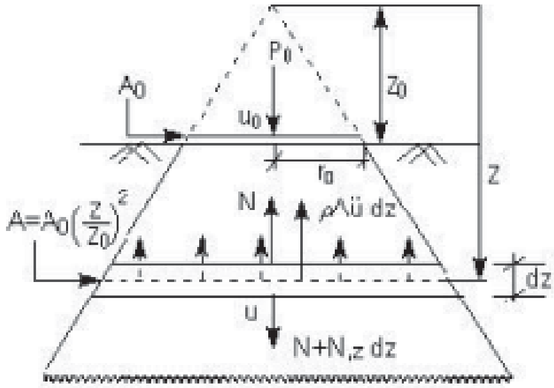

Figure 2. Wave propagation in semi-infinite truncated cone under vertical translational load

The area A at depth z equals, $A = (z^2 / z_0^2) A_0$ with $A_0 = ðr_0^2$, where z is measured from the apex. With c denoting the appropriate wave velocity of compression-extension waves (dilatational waves) and ñ the mass density, $ñc^2$ is equal to corresponding elastic modulus (constrained modulus). Also, u represents the axial displacement and N the axial force. Radial effects are disregarded. Formulating the equilibrium equation of an infinitesimal element (Fig. 2) taking the inertial loads into account,

$$-N + N + N_{,z}\,dz - \rho A\,dz\,\ddot{u} = 0 \qquad (5)$$

and substituting the force-displacement relationship,

$$N = \rho c^2 A u_{,z} \qquad (6)$$

leads to the equation of motion in time domain of translational cone

$$u_{,zz} + \frac{2}{z} u_{,z} - \frac{\ddot{u}}{c^2} = 0 \qquad (7)$$

which may be written as one-dimensional wave equation in zu

$$(zu)_{,zz} - \frac{1}{c^2}(z\ddot{u}) = 0 \qquad (8)$$

The familiar solution of the wave equation in time domain is

$$zu = z_0 f\left(t - \frac{z - z_0}{c}\right) + z_0 g\left(t + \frac{z - z_0}{c}\right) \qquad (9)$$

Where, f and g are arbitrary functions of the argument for outward propagating waves (in the positive z-direction) and inward propagating waves (in the negative z-direction), respectively. In a radiation problem like footing vibration, only outward propagating waves are permissible (g = 0). Thus the above equation is reduced to

$$u(z,t) = \frac{z_0}{z} f\left(t - \frac{z - z_0}{c}\right) \qquad (10)$$

In frequency domain Eq. (10) can be written as

$$u(\omega) = \frac{z_0}{z} e^{-i\omega\left(\frac{z - z_0}{c}\right)} \qquad (11)$$

A. Dynamic Impedance

For a prescribed displacement u_0 of the disk, the interaction force P_0 is calculated as follows. Enforcing boundary condition u ($z = z_0$) = u_0 to Eq. (10) yields

$$u_0(t) = f(t) \qquad (12)$$

Also, $$P_0 = -N(z = z_0) = -\rho c^2 A_0 u_{0,z} \qquad (13)$$

Differentiating Eq. (10) and substituting its value at $z = z_0$ in Eq. (13), we get

$$P_0(t) = \frac{\rho c^2 A_0}{z_0} u_0(t) + \rho c A_0 \dot{u}_0(t) \qquad (14)$$

Eq. (14) can be written in the form

$$P_0(t) = K u_0(t) + C \dot{u}_0(t) \qquad (15)$$

In this interaction force-displacement relationship K and C are constant coefficients of spring and dashpot:

$$K = \frac{\rho c^2 A_0}{z_0} \quad \text{and} \quad C = \rho c A_0 \qquad (16)$$

This frequency independent spring coefficient (static stiffness) K is equated to the exact static stiffness in order to get the value of the aspect ratio of the cone model. Thus

$$\frac{\rho c^2(\pi r_0^2)}{z_0} = \frac{4Gr_0}{1-\nu} \quad \text{or} \quad \frac{z_0}{r_0} = \frac{\pi}{4}(1-\nu)\left(\frac{c}{c_s}\right)^2 \qquad (17)$$

The Eqs. (14) to (16) are valid for compressible soil i.e. õd”1/3. For incompressible soil (1/3 < õ d” 1/2), the concept of introducing trapped mass is enforced. Thus,

$$P_0(t) = K u_0(t) + C \dot{u}_0(t) + \Delta M \ddot{u}_0(t) \qquad (18)$$

Where, $\Delta M = \mu \rho \pi r_0^3$, the trapped mass.

The discrete element model is shown in Fig. 3. The force-displacement relationship is given by

$$P_0(\omega) = (K - \omega^2 \Delta M + i\omega C) u_0 \qquad (19)$$

which results with the dynamic impedance

$$\bar{K}(\omega) = K - \omega^2 \Delta M + i\omega C \qquad (20)$$

Using dimensionless frequency parameter a_0 defined by $a_0 = \omega r_0 / c_s$ in Eq. (20), $\bar{K}(\omega)$ is transformed to

$$\bar{K}(a_0) = K[k(a_0) + i a_0 c(a_0)] \qquad (21)$$

Where, $k(a_0) = 1 - \frac{\mu}{\pi}\frac{z_0}{r_0}\frac{c_s^2}{c^2} a_0^2$, $c(a_0) = \frac{z_0}{r_0}\frac{c_s}{c}$, with trapped mass coefficient, μ = 0, for $\nu \le 1/3$ and $\mu = 2.4\pi(\nu - \frac{1}{3})$, for $1/3 < \nu \le 1/2$

I. Results And Discussions

A simplified analytical model has been developed to study the vertical dynamic response of a foundation resting on homogenous elastic half-space. Results have been computed using a simple computer code, varying the influencing parameters widely. To verify the applicability of the model in practice, rigorous comparisons of computed results viz., stiffness and damping coefficients, frequency-amplitude response, frequency-magnification response and phase angles of vertical displacement response have been made with the reported semi-analytical and analytical models.

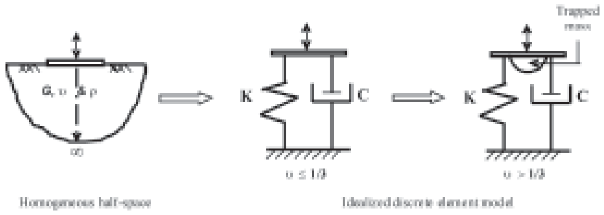

Figure 3. Discrete element model for foundation vibration on homogeneous half=space

A. Comparison of Dynamic Impedance Functions

The spring coefficient and damping coefficient of a rigid massless circular foundation resting on homogenous elastic half-space with Poisson ratio í = 0.45 are computed using cone model and compared with the reported results of Luco and Mita (1987) in Fig.4. The comparison shows a good agreement in the stiffness coefficient in the range of frequency compared, but the cone model predicts a little higher damping at lower frequency and lower damping at higher frequency. Though the cone model for half-space gives frequency independent stiffness and damping coefficient for í 1/3, introduction of a trapped mass for 1/3< 1/2 under vertical mode is justified as this results in a downward parabolic stiffness coefficient (Fig. 4), thus matching well with the published results based on more rigorous analysis.

B. Comparison of Frequency-Amplitude and Frequency-Magnification Response

The computed frequency-amplitude response of a rigid circular massive foundation resting on homogeneous elastic half-space is compared with the available analytical solutions of Richart et al. (1970) for four different mass ratios and for both constant force and frequency dependent force excitation under vertical mode [Fig 5(a) & (b)]. From these figures it is observed that though there is very good agreement with regard to resonance frequency, at resonance relatively less values of amplitudes (21 to 25% less) are predicted by the cone model. A maximum of 25% decrease in peak amplitude is observed for the case with highest mass ratio (b_0 = 40).

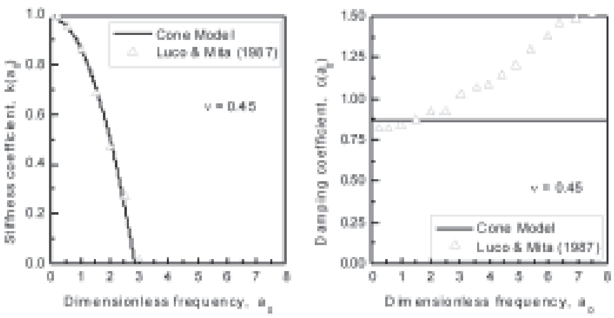

Figure 4. Comparison of dynamic impedance functions for õ = 0.45

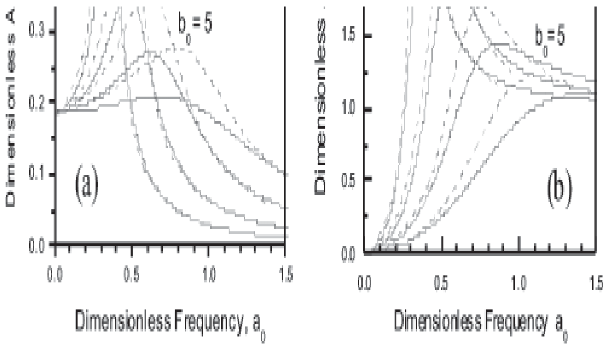

Figure 5. Comparison of frequency-amplitude response (n = 0.25): (a) Constant force excitation; (b) Frequency dependent excitation

The computed frequency-magnification response of a rigid circular foundation is also compared with the analytical solutions of recently developed optimal equivalent model of Chen & Shi (2006) for different mass ratios (b_0 = 1, 5, 10 & 30) at two values of Poisons ratio i.e. õ = 1/3 & 0.5 [Fig. 6 & Fig. 7]. From these figures it is observed that for õ = 0.5 (incompressible soil) there is excellent agreement with regard to both resonance frequency and peak amplitude. For õ = 1/3, the model predicts an accurate value of resonant frequency and slightly less value of peak amplitude, indicating little higher damping, particularly at higher mass ratio i.e. at b_0 = 30.

C. Comparison of Phase Angles

In the problems of vertical foundation vibrations, the accuracy of cone model on evaluating the stiffness, damping, frequency-amplitude response and frequency-magnification response is well investigated in previous sections. The secondary important response for the foundation, phase angles of the vertical displacement response, which is computed and compared with the optimal equivalent model of Chen and Shi (2006) as shown in Fig. 8. From Fig. 8, it is observed that the results obtained by cone model agree with the solutions based on optimal equivalent model of Chen and Shi (2006) well in low frequencies, which is very important for practical application, though little differences are observed in the high frequencies. The differences are reduced rapidly as the Poisson's ratio of soil increases. Moreover, the magnification factors in the high frequencies are much smaller than that in the low frequencies. Hence, the accuracy of cone model for evaluating the phase angles of foundation responses are deemed accurate enough for engineering applications.

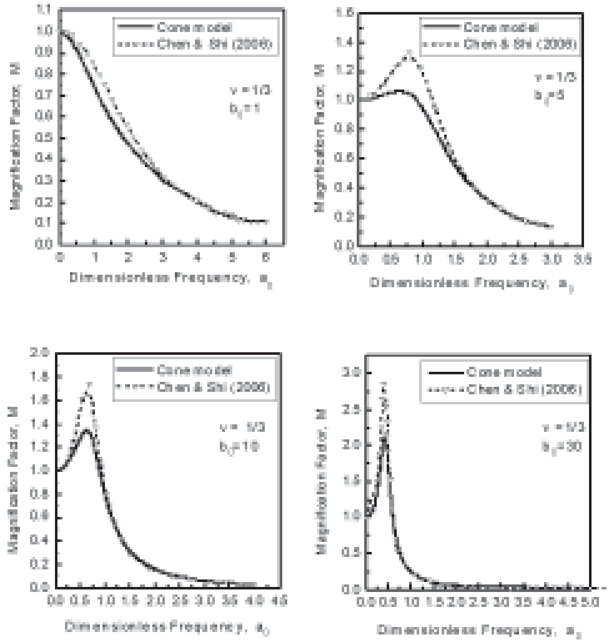

Figure 6. Comparison of dimensionless frequency-amplitude response for υ = 1/3 and mass ratios 1, 5, 10 and 30.

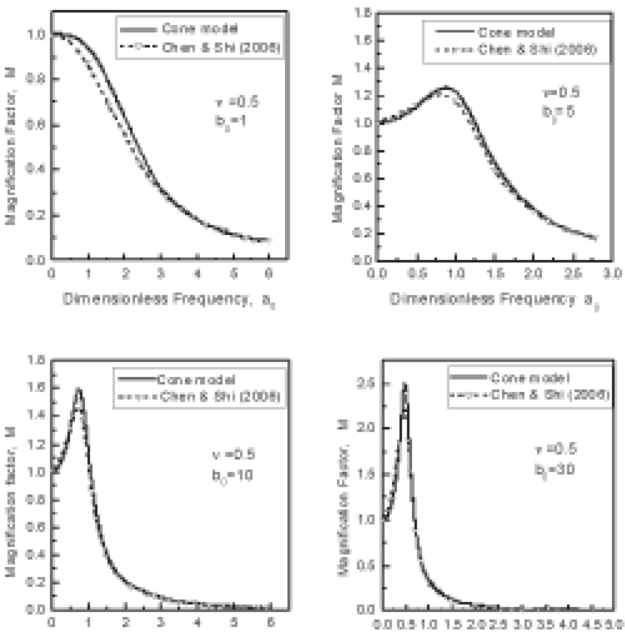

Figure 7. Comparison of dimensionless frequency-amplitude response for υ = 0.5 and mass ratios 1, 5, 10 and 30.

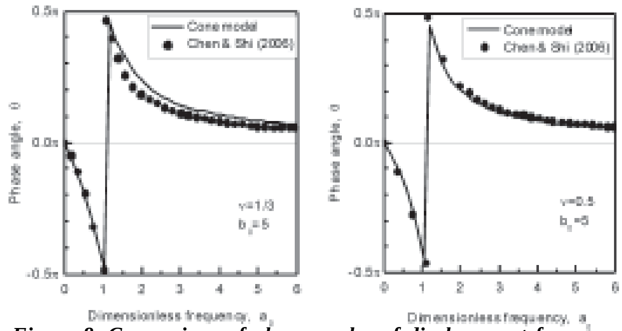

Figure 8. Comparison of phase angles of displacement for mass ratio 5 and õ = 1/3 & 5

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