The equivalence of real and reduced-mass systems

Kepler provided a **kinematic** description of orbits: his laws describe the shape of a planet's motion around the Sun, and allow one to calculate its position as a function of time. In a way, Kepler did for orbital motion what Galileo did for projectile motion; he put it in mathematical terms. But what Kepler (and Galileo) did not do was to explain **why.** Why should orbits be ellipses? Why should a planet move faster when it is closer to the Sun?

Newton answered the "why" by setting forth a few simple rules of **dynamics:** the interaction between objects, their motion, and the forces which act upon them. Recall Newton's three laws of motion:

1. A body moves in with constant velocity unless acted upon by an external force
2. If pushed by a force **F**, a body accelerates such that
3.
4. F = m * a
5. If body 1 exerts a force on body 2, then body 2 exerts an equal and opposite force on body 1

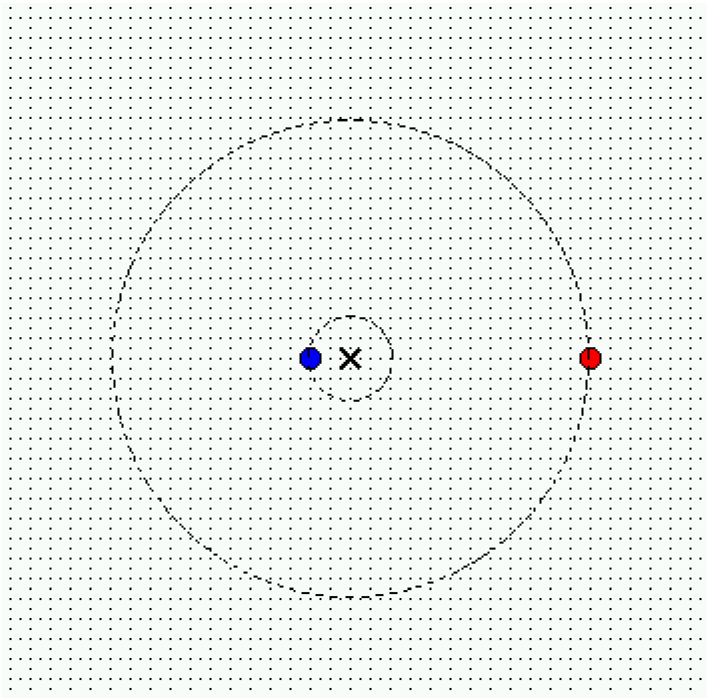
To these rules, Newton added one more which was especially useful in the celestial realm, where the dominant force between objects is gravity.

4. The gravitational force between two objects is attractive, directed along the line joining the bodies, and of magnitude
5.
6. m1 * m2
7. F = G -------
8. r^2

It is possible to start with Newton's postulates and derive Kepler's three laws; see Section 2.3 of your textbook. Let's keep Newton's laws in mind as we examine the properties of real and reduced-mass orbits: forces, energies, angular momenta.

Gravitational force

Look at the example I handed out in class last week: two masses, **m1** = **6** and **m1** = **1**, moving around their common center of mass in circular orbits.



Consider the gravitational force between these two bodies. Please express all your answers to the questions below so that they are in terms of the small mass **m2** and its distance **r2** away from the center of mass.

1. What is the gravitational force between the two bodies?

One can calculate the equivalent reduced-mass system, in which a small mass **μ** orbits around a stationary large mass **M** = **m1** + **m2.** For the reduced-mass case,

2. What is the gravitational force between the two bodies?

Apparently, the reduced-mass orbit has the same gravitational force as the real orbit. How nice.

Kinetic energy

The kinetic energy of a system is simply the sum over all the bodies of

1 2

$$KE = \frac{\quad}{2} m * v$$

Again look at the example with mass ratio 6:1 and circular orbits.

Suppose that the period of the orbit is **P**. Please express all your answers to the questions below so that they are in terms of the small mass **m2** and its distance **r2** away from the center of mass.

1. What is the kinetic energy of body 1 in the real system?
2. What is the kinetic energy of body 2 in the real system?
3. What is the total kinetic energy in the real system?

One can calculate the equivalent reduced-mass system, in which a small mass **μ** orbits around a stationary large mass **M = m1 + m2**. For the reduced-mass case,

4. What is the kinetic energy of the large mass **M**?
5. What is the kinetic energy of the reduced mass **μ**?
6. What is the total kinetic energy of the reduced-mass system?

Apparently, the reduced-mass orbit has the same total kinetic energy as the real orbit. How nice.

Gravitational potential energy

The gravitational potential energy of a system is the sum over all PAIRS of bodies of

$$GPE = - G \frac{m1 * m2}{r}$$

Consider again the example of two bodies with mass ratio 6:1. Please express all your answers to the questions below so that they are in terms of the small mass **m2** and its distance **r2** away from the center of mass.

1. What is the gravitational potential energy of the real system?

One can calculate the equivalent reduced-mass system, in which a small mass **μ** orbits around a stationary large mass **M = m1 + m2**. For the reduced-mass case,

2. What is the gravitational potential energy of the equivalent reduced-mass system?

Apparently, the reduced-mass orbit has the same gravitational potential energy as the real orbit.

Angular momentum

The angular momentum of a body in a circular orbit is

$$L = m * v * r$$

Consider again the example of two bodies with mass ratio 6:1. Please express all your answers to the questions below so that they are in terms of the small mass **m2** and its distance **r2** away from the center of mass.

1. What is the angular momentum of large body **m1**?
2. What is the angular momentum of small body **m2**?
3. What is the total angular momentum of the real system?

One can calculate the equivalent reduced-mass system, in which a small mass **μ** orbits around a stationary large mass **M = m1 + m2**. For the reduced-mass case,

4. What is the angular momentum of total mass **M**?
5. What is the angular momentum of reduced-mass **μ**?
6. What is the total angular momentum of the reduced-mass system?

Apparently, the reduced-mass orbit has the same angular momentum as the real orbit.

The bottom line is that many of the interesting physical properties of the REAL orbital system turn out to be identical to the computed properties of the REDUCED-MASS system. That suggests that we can

- make the simple **(r, θ)** relative measurements of a binary star's orbit
- (plus the extra measurements required to determine the relative masses)
- use the relative measurements, plus the reduced-mass formalism, to compute energy, angular momentum, total mass, etc.

In other words, it gives us confidence that we can use the measurements to derive the quantities of interest -- especially the mass of the stars.

Source: http://spiff.rit.edu/classes/phys440/lectures/equiv/equiv.html