This lecture purports to the macroscopic Maxwell's equations in the differential forms and their revealation about the propagation of light in vacuum and in matter. Microscopic Maxwell's equations and their relationship with the macroscopic Maxwell's equations are also discussed.

1 Macroscopic Maxwell's equations

In a macroscopic medium, let $\mathbf{E}$ denote the macroscopic electric field strength, $\mathbf{D}$-the electric displacement, $\mathbf{H}$-the magnetic field strength, $\mathbf{P}$-the electric polarization of the macroscopic medium, $\mathbf{M}$-the magnetization of the macroscopic medium, $\mathbf{B}$-the magnetic induction, ρ-the charge density and $\mathbf{J}$-the electric current density. In the macroscopic medium, the charge and current densities are continuous functions of spatial dimensions.

It is well known fact that the sources for the electric displacement are free charges. Thus one can write

$$\nabla.\mathbf{D} = \rho. \tag{1.1}$$

The relation

$$\nabla.\mathbf{B} = 0, \tag{1.2}$$

tells that the magnetic induction $\mathbf{B}$ does not have any source. The fact that temporally varying electric and magnetic fields, $\mathbf{E}$ and $\mathbf{H}$ are responsible for the generation of each other and that the macroscopic current density $\mathbf{J}$ is responsible for the generation of the magnetic field strength $\mathbf{H}$ can be represented by the following respective relations,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1.3}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}. \tag{1.4}$$

Eqs. (**??**) to (1.4) form the macroscopic Maxwell's equations in differential form and constitute the basis of all classical electromagnetic phenomena. The electromagnetic properties of the macroscopic medium are furnished by the macroscopic material related equations given by

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \quad \text{and} \tag{1.5}$$

$$\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M}, \tag{1.6}$$

where ε_0 is the permittivity of vacuum and μ_0 is the permeability of vacuum. Eqs. (1.5) and (1.6) are always valid as they do not impose any conditions on the macroscopic medium.

On applying divergence to both sides of Eq. (1.4), one obtains the continuity equation for the electric charges given as

$$\nabla.\mathbf{J} + \frac{\partial \rho}{\partial t} = 0. \tag{1.7}$$

In the presence of electromagnetic fields, the force $\mathbf{F}$ acting on a point charge q is given by the Lorentz force equation

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{1.8}$$

If **F** is the net force acting on a body of mass m and if **a** is the acceleration of the body along the same direction as **F**, then, according to Newton's second law of motion, one can write,

$$\mathbf{F} = m\mathbf{a}. \tag{1.9}$$

The macroscopic Maxwell's equations, when combined with the Lorentz force equation given by (1.8) and Newton's second law of motion, given by Eq. (1.9), give the dispersion of the classical dynamics of the various charge and particles that are interacting with each other and the electromagnetic fields in the most complete manner.

2 Electromagnetic radiation in vacuum

Vacuum is represented by the following conditons:

$$\mathbf{P} = \Phi, \quad \mathbf{M} = \Phi, \quad \rho = 0 \quad \text{and} \quad \mathbf{J} = \Phi, \tag{2.1}$$

where Φ represents a null-vector.

On substituting these into the Maxwell's equations given by Eqs (**??**)-(1.4) and on making suitable mathematical manipulations, one finally obtains the plane wave equation for the electric field as

$$\nabla^2 \mathbf{E}(\mathbf{r},t) = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}(\mathbf{r},t)}{\partial t^2}. \tag{2.2}$$

The plane wave solution for Eq. (2.2) has the form

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 \exp[i(\mathbf{k}.\mathbf{r} - \omega t)], \tag{2.3}$$

where $\mathbf{E}_0$ is the amplitude, **k**- the wave vector and ω-the corresponding angular frequency. On substituting Eq. (2.3) into Eq. (2.2), with $k = |\mathbf{k}| = \frac{2\pi}{\lambda}$ and with speed of light in vacuum, $c \equiv \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$, one obtains the relation $\omega = ck$.

For all waves, the phase velocity v_{ph} which is defined as the velocity with which a certain phase propagates, is defined as $v_{ph} = \frac{\omega}{k}$. Group velocity v_g is defined as the velocity of the center of mass of a wave packet with central frequency ω and covering a small frequency interval $d\omega$ and is represented as $v_g \equiv \frac{\partial \omega}{\partial k}$. In vacuum, $v_{ph} = v_g = C = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$. Maxwell's equations impose certain constraints on **E**, **D**, **B**, **H** and **k**, which are as follows:

1. The electromagnetic wave is transverse in **E** and **B**
2. The electric and magnetic fields are perpendicular to each other, i.e, $\mathbf{E} \perp \mathbf{B}$; $\mathbf{D} \perp \mathbf{H} \perp \mathbf{k}$.
3. Also, in vacuum, $\mathbf{E} \parallel \mathbf{D}$ and $\mathbf{H} \parallel \mathbf{B}$

3 Electromagnetic radiation in matter: Linear optics

In matter, if one assumes that there are no macroscopic free space charges, such that $\rho = 0$, and also that it is a non-magnetic material such that $\mathbf{M} = 0$, the plane wave equation takes the form,

$$\nabla^2 \mathbf{E} = \mu_0 \left[\frac{\partial \mathbf{J}}{\partial t} + \frac{\partial^2 \mathbf{P}}{\partial t^2} + \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \right]. \tag{3.1}$$

Further as $\mathbf{J} = \sigma\mathbf{E}$, where σ is the electric conductivity, if one assumes that it is a weakly or intrinsic doped semiconductor, then the term $\frac{\partial \mathbf{J}}{\partial t}$ can be safely neglected from Eq. (3.1). Hence one can write

$$\nabla^2\mathbf{E} = \mu_0\left[\frac{\partial^2\mathbf{P}}{\partial t^2} + \varepsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2}\right]. \tag{3.2}$$

If a linear relation exists between **P** and **E** such that

$$\mathbf{P} = \varepsilon_0\chi_e\mathbf{E} \tag{3.3}$$

or

$$\begin{aligned} \mathbf{D} &= \varepsilon_0(1+\chi_e)\mathbf{E}, \\ &= \varepsilon_0\varepsilon(\omega)\mathbf{E}, \end{aligned} \tag{3.4}$$

where $\varepsilon(\omega) = H\chi_e$ with χ_e being the electric susceptibility of the medium, Eq. (3.2) can be written as

$$\nabla^2\mathbf{E} = \mu_0\varepsilon_0\varepsilon(\omega)\frac{\partial^2\mathbf{E}}{\partial t^2}. \tag{3.5}$$

The plane wave solution to Eq. (3.5) takes the form

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0\exp[i(\mathbf{k}.\mathbf{r} - \omega t)]. \tag{3.6}$$

On substituting Eq. (3.6) into Eq. (3.5), one obtains

$$\varepsilon(\omega) = \frac{c^2k^2}{\omega^2}, \tag{3.7}$$

known as the polariton equation. Eq. (3.7) can also be written as

$$k = \frac{\omega}{c}\sqrt{\varepsilon(\omega)}. \tag{3.8}$$

Eq. (3.8) suggests that $\varepsilon(\omega)$ and hence k can become complex.

Let

$$\sqrt{\varepsilon(\omega)} = n(\omega) + i\kappa(\omega), \tag{3.9}$$

where $n(\omega)$ and $\kappa(\omega)$ are real.

$$k = \frac{\omega}{c}n(\omega) + i\frac{\omega}{c}\kappa(\omega). \tag{3.10}$$

Thus one can say that in vacuum, the wave vector **k**, along whose direction an electromagnetic wave propagates, is real. In matter, light can propagate with a complex wave vector **k**. The oscillatory part of the electromagnetic wave is described by the direction of real part of the wave vector **k** and is parallel to $\mathbf{D}\times\mathbf{B}$.

The term $n(\omega)$ in Eq. (3.10) is termed as the refractive index of the macroscopic medium. It represents the oscillatory spatial propagation of light in matter.

4 Constitutive relations

In a given macroscopic medium, the currents and charges generate the electric and magnetic fields defined by the macroscopic Maxwell's equations. But Maxwell's equations do not provide the answer for how these currents and charges originate in the given macroscopic medium. Hence Maxwell's equations have to be supplemented by certain governing relations, that provide the answer for the behaviour of matter under the influence of the electromagnetic fields. The above mentioned governing relations are known as the material constitutive relations.

4.1 Non-dispersive, linear and isotropic medium

In a dispersive medium, the phase velocity of the group velocity of the electromagnetic wave depends on the frequency. A macroscopic medium where the velocity is frequency independent is called a non-dispersive medium. In a linear medium, there exists a linear relationship between the polarization and the electric field strength. An isotropic medium is one where the permittivity ε and the permeability μ of the medium are same and uniform in all directions. Let χ_e be the electric susceptibility and χ_m the magnetic susceptibility for a non-dispersive linear and isotropic medium. Then the corresponding constitutive relations take the form,

$$\begin{aligned} \mathbf{P} &= \varepsilon_0 \chi_e \mathbf{E}, \\ \mathbf{M} &= \chi_n \mathbf{H}, \\ \mathbf{D} &= \varepsilon_0 \varepsilon \mathbf{E}, \\ \mathbf{B} &= \mu_0 \mu \mathbf{H} \quad \text{and} \\ \mathbf{J}_c &= \sigma \mathbf{E}, \end{aligned} \tag{4.1}$$

where ε is the permittivity and μ is the permeability of the medium.

4.2 Nonlinear medium

If there exists a nonlinear relationship between the electric polarization $\mathbf{P}$ and electric field strength $\mathbf{E}$, the medium is said to be nonlinear. In a nonlinear medium new frequencies are generated in the form of either sum frequency generation or difference frequency generation of higher order harmonic generation. In that case, the electric susceptibilities, denoted as $\chi_e^{(n)}$ are no longer scalars or vectors but are complex tensors. Also, they are functions that constitute the material, various frequencies and last but not the least depend on all other properties of the light waves that are incident on the medium. A j^{th} order suceptibility tensor is of rank $j+1$. In general, the nonlinear relationship between the polarization $\mathbf{P}$ and the electric field strength $\mathbf{E}$ can be written as follows:

$$\mathbf{P} = \varepsilon_0 (\chi_e^{(1)} . \mathbf{E} + \chi_e^{(2)} : \mathbf{EE} + \chi_e^{(3)} : \mathbf{EEE} + \ldots), \tag{4.2}$$

where ε_0 is the permittivity in free space.

$\chi_e^{(j)}$ is a j^{th} order susceptibility tensor of rank $j+1$. The effect of $\chi_e^{(1)}$ is included in the refraction index n. $\chi_e^{(2)}$ is instrumental in initiating second-harmonic generation, sum frequency generation, etc. $\chi_e^{(S)}$ introduces nonlinear effects such as Kerr nonlinearity, four wave mixing, third harmonic generation, etc. The total induced polarization $\mathbf{P}$ can be written as

$$\mathbf{P}(\mathbf{r},t) = \mathbf{P}_L(\mathbf{r},t) + \mathbf{P}_{NL}(\mathbf{r},t), \tag{4.3}$$

where

$$\mathbf{P}_L = \varepsilon_0 \int_{-a}^{t} \chi^{(1)}(\tau - \tau').\mathbf{E}(\mathbf{r},\tau')d\tau' \tag{4.4}$$

For second order nonlinearity, $\mathbf{P}_{NL}$ can be written as

$$\mathbf{P}_{NL} = \varepsilon_0 \int_{-\infty}^{t}\int_{-\infty}^{t}\int_{-\infty}^{t} \chi^{(2)}(\tau-\tau_1,\tau-\tau_2) : \mathbf{E}(\mathbf{r},\tau_1)\mathbf{E}(\mathbf{r},\tau_2)d\tau_1 d\tau_2 \tag{4.5}$$

and so on. The other constituent relations can be written as

$$\mathbf{D} = \varepsilon_0\mathbf{E} + (\mathbf{P}_L + \mathbf{P}_{NL}) \tag{4.6}$$

and

$$\mathbf{B} = \mu_0\mathbf{H} + \mathbf{M}. \tag{4.7}$$

4.3 anisotropic medium

The macroscopic medium is said to be anisotropic if ε and μ are not uniform in all directions. In that case, ε and μ take tensorial forms.

4.4 Inhomogeneous medium

If the material parameters ε, μ and σ are functions of spatial dimensions, the medium is said to be inhomogeneous.

5 Microscopic Maxwell's equations and their revelation

The macroscopic fields do not explain the microscopic nature of matter as they are spatial average of over the microscopic fields associated with discrete chareges. As nanoscience requires the representation of fields on an atomic or sub-atomic scale, microscopic Maxwell's equations are required as they represent the entire matter to be made up of charged and uncharged particles.

If $\rho(\mathbf{r},t)$ and $\mathbf{J}(\mathbf{r},t)$ are the charge density and current density in the microscopic medium, which are written as spatial parameters, the microscopic Maxwell's equations cna be written in the differential form for the point charges as,

$$\nabla.\mathbf{E}_{micro}(\mathbf{r},t) = \frac{1}{\varepsilon_0}[\rho_0(\mathbf{r},t) + \rho_{mat}(\mathbf{r},t)], \tag{5.1}$$

$$\nabla.\mathbf{B}_{micro}(\mathbf{r},t) = 0, \tag{5.2}$$

$$\nabla \times \mathbf{E}_{micro}(\mathbf{r},t) = -\frac{\partial \mathbf{B}_{micro}(\mathbf{r},t)}{\partial t}, \tag{5.3}$$

$$\nabla \times \mathbf{B}_{micro}(\mathbf{r},t) = \frac{1}{c^2}\frac{\partial E_{micro}(\mathbf{r},t)}{\partial t} + \mu_0[\mathbf{J}+0(\mathbf{r},t)+\mathbf{J}_{mat}(\mathbf{r},t)]. \tag{5.4}$$

There are no corresponding $\mathbf{D}_{micro}(\mathbf{r},t)$ and $\mathbf{H}_{micro}(\mathbf{r},t)$ as all the point charges are included in $\rho_{mat}(\mathbf{r},t)$ and $\mathbf{J}_{mat}(\mathbf{r},t)$. $\rho_{mat}(\mathbf{r},t)$ is defined as the polarization charge density and hence can be written as,

$$\nabla.\mathbf{P}_{micro}(\mathbf{r},t) = -\rho_{mat}(\mathbf{r},t). \tag{5.5}$$

$$\mathbf{J}_{mat}(\mathbf{r},t = (\mathbf{J}_{mat}(\mathbf{r},t) - \mathbf{J}_P) + \mathbf{J}_P. \tag{5.6}$$

Therefore,

$$\nabla.\mathbf{J}_{mat}(\mathbf{r},t) + \frac{\partial \rho_{mat}(\mathbf{r},t)}{\partial t} = 0. \tag{5.7}$$

Also

$$\nabla.(\mathbf{J}_{mat}(\mathbf{r},t) - \mathbf{J}_P(\mathbf{r},t)) = 0. \tag{5.8}$$

6 Additional reading and references

1. D. J. Griffiths, Introduction to Electrodynamics (Addison-Wesley, New York, 2012).

Source:
http://nptel.ac.in/courses/118106021/2