

MHD flow of radiating and chemically reacting viscoelastic fluid through a porous medium in porous vertical channel with constant suction

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ABSTRACT: An investigation of incompressible, electrically conducting, chemically reacting and radiating viscoelastic fluid through porous medium with constant injection/suction velocity in the presence of uniform magnetic field applied perpendicular to the plane of the plates of the channel is carried out. A closed form solutions for the equations governing the flow velocity, temperature and concentration profile are obtained. The effect of various parameters on velocity, temperature, concentration, skin friction, rate of heat and mass transfer at the plate are evaluated numerically and expressed graphically for different values of flow parameters.

Keywords: Constant suction, chemically reacting, MHD, radiating and viscoelastic.

I. INTRODUCTION

During the past few decades, there has been a growing interest in non-Newtonian fluids. The flows of non-Newtonian fluids are found in a variety of applications: from drilling oil, and gas wells and well completion operations to the industrial processes involving a waste fluid, synthetic fibre, food stuff and the exertion of molten plastic as well as in some flows of polymer solutions. The large variety of fluids and its industrial application has been a major motivation for research in non-Newtonian flows.

Flows through porous media are frequently used in filtering of gasses, liquid and drying of bulk materials. In electrochemical engineering, porous electrodes and permeable, semi permeable diaphragms are used to obtain improved current efficiencies. In the field of agricultural engineering, porous media heat transfer plays an important role particularly in germinations of seeds. Above all man does a part of his breathing through his porous skin. Several researchers have studied the two dimensional free convection, heat and mass transfer flow of an elastico- viscous fluid through porous medium. The representative study in this area can be found in [1-4].

Chemical reactions usually accompany a large amount of exothermic and endothermic reactions. These characteristics can be easily seen in a lot of industrial processes. Recently, it has been realized that it is not always permissible to neglect the convection effects in porous constructed chemical reactors. The reaction produced in a porous medium was extraordinarily in common, such as the topic of PEM (Polymer Electrolyte Membrane) fuel cells modules and the polluted underground water because of discharging the toxic substance etc. The effect of chemical reaction on various problems can be found in [5-8]

The role of thermal radiation is of major importance in the design of many advanced energy convection systems operating at high temperature and knowledge of radiative heat transfer becomes very important in nuclear power plants, gas turbines and the various propulsion devices for aircraft, missiles and space vehicles. Grosan and Pop [9], Joshi and Kumar [10] and Srinivas and Muthuraj [11] have made investigations of fluid flow with thermal radiation. Ghosh [12] investigated the hydrodynamic fluctuating flow of a viscoelastic fluid in a porous channel, where the channels oscillate with a given velocity in their own planes. Das [13] has analysed viscoelastic effects on unsteady two-dimensional and mass transfer of a viscoelastic fluid in a porous channel with radiative heat transfer.

Flow through the porous media with heat and mass transfer are seen in wide range of applications such as solar power collector, chemical catalytic reactor, insulation of chemical reactor, fluid film lubrication and analysis of polymer in chemical engineering. Representative studies in this area can be found in [14-16].

Motivated by the above researches, the present study is aimed to investigate the flow of incompressible electrically conducting, chemically reacting and radiating viscoelastic fluid through porous medium in vertical porous channel with constant suction in the presence of uniformly applied magnetic field.

II. FORMULATION OF THE PROBLEM

Consider an oscillatory viscoelastic, incompressible and electrically conducting fluid through a highly porous medium bounded between two infinite vertical porous plates at distance ' d ' apart, with constant injection velocity is applied at the left plate and same suction velocity at the right plate the channel. We introduce a Cartesian co-ordinate system with X^* - axis oriented vertically upward of this channel and Y^* - axis taken perpendicular to the planes of the plates. A uniform magnetic field with magnetic flux density vector B_0 is applied perpendicular to the plane of plates. Since the plate of the channel occupying the planes $y = \pm \frac{d}{2}$ are of infinite extent, all the physical quantities depends on y and t only. The equation of continuity $\nabla \cdot V$ on integration gives $v^* = 0$ for non-porous plates. Denote the velocity $V = (u, 0)$ in x^*, y^* direction respectively. The schematic diagram of the physical problem is shown in the Figure 1.

The flow is governed by following assumption

- 1 A uniform magnetic field is applied normal to the planes of the plates.
- 2 Boussinesq approximation is applied.
- 3 The magnetic Reynolds number is assumed to be small enough so that the induced magnetic field is negligible.
- 4 The effect of viscous and Joule's dissipation is assumed to be negligible in the energy equation as small velocity is usually encountered in free convection flows.
- 5 There exists a first order chemical reaction between the fluid and species concentration.
- 6 The level of species concentration is very low so that the heat generated during chemical reaction can be neglected.

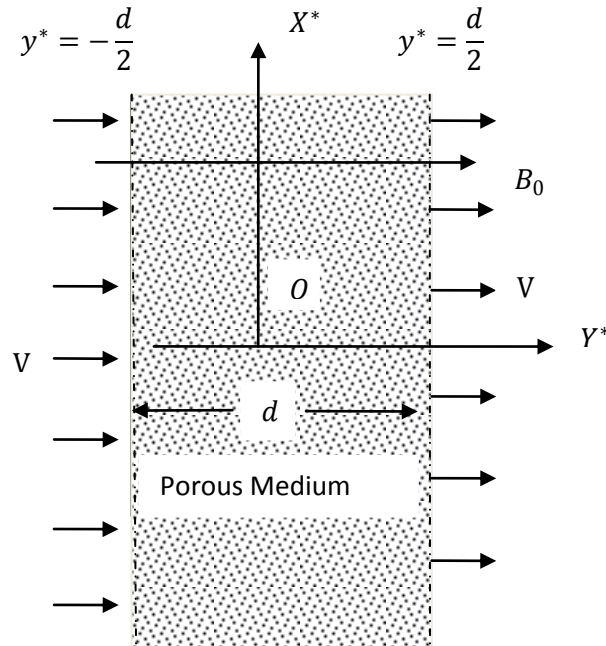


Figure1. The Geometrical configuration of the physical problem

Under the above assumption the flow is governed by the following equations

$$\frac{\partial u^*}{\partial t^*} + V \frac{\partial u^*}{\partial y^*} = -\frac{1}{\rho} \frac{\partial p^*}{\partial x^*} + \frac{1}{\rho} \frac{\partial \tau_{x^*y^*}^*}{\partial y^*} + g\beta T^* + g\beta_c C^* - \frac{\sigma B_0^2}{\rho} u^* - \frac{\nu}{K^*} u^* \quad (1)$$

where $\tau_{x^*y^*}^*$ are the component of shear stress of the viscoelastic fluid.

$$\tau_{x^*y^*}^* = \mu \frac{\partial u^*}{\partial y^*} - \frac{\mu}{\alpha} \frac{\partial \tau_{x^*y^*}^*}{\partial t^*}$$

where μ is the coefficient of viscosity and α is the modulus of rigidity. If the limit α tends to infinity or at the steady state, the fluid behaves like a viscous fluid without elasticity. Solving equation for $\tau_{x^*y^*}^*$ in terms of velocity u we obtain

$$\frac{\partial \tau_{x^*y^*}^*}{\partial y^*} = \frac{\partial}{\partial y^*} \left(\mu \frac{\partial u^*}{\partial y^*} \right) - \frac{1}{\alpha} \frac{\partial}{\partial y^*} \left(\mu \frac{\partial}{\partial t^*} \left(\mu \frac{\partial u^*}{\partial y^*} \right) \right)$$

where the terms $\frac{1}{\alpha} \frac{\partial}{\partial y^*} \left(\mu \frac{\partial}{\partial t^*} \left(\mu \frac{\partial u^*}{\partial y^*} \right) \right)$ which are proportional to $\frac{1}{\alpha^2}$ have been neglected substituting $\frac{\partial \tau_{x^*y^*}^*}{\partial y^*}$ in the momentum equation yields following Navier- stokes equation

$$\frac{\partial u^*}{\partial t^*} + V \frac{\partial u^*}{\partial y^*} = -\frac{1}{\rho} \frac{\partial P^*}{\partial x^*} + \nu \frac{\partial^2 u^*}{\partial y^{*2}} + k_0 \frac{\partial^3 u^*}{\partial t^* \partial y^{*2}} + g\beta T^* + g\beta_c C^* - \frac{\sigma B_0^2}{\rho} u^* - \frac{\nu}{K^*} u^* \quad (2)$$

$$\frac{\partial T^*}{\partial t^*} + V \frac{\partial T^*}{\partial y^*} = \frac{k}{\rho C_p} \frac{\partial^2 T^*}{\partial y^{*2}} - \frac{1}{\rho C_p} \frac{\partial q}{\partial y^*} \quad (3)$$

$$\frac{\partial C^*}{\partial t^*} + V \frac{\partial C^*}{\partial y^*} = D \frac{\partial^2 C^*}{\partial y^{*2}} - D_1 C^* \quad (4)$$

where ρ is density, P^* is the modified pressure gradient, ν is the kinematics viscosity, g is the acceleration due to gravity, σ is the electrical conductivity, β is coefficient of volume expansion, β_c is the coefficient of expansion with concentration, $K_0 = \frac{\mu^2}{\rho\alpha}$ = viscoelasticity, C_p is specific heat at constant pressure, κ is the thermal conductivity, D is the molecular diffusivity and D_1 reaction rate constant.

Following Cogley et al [17] the last term in the energy equation stand for the radiative heat flux which is given by

$$\frac{\partial q}{\partial y^*} = 4a^2 T^* \quad (5)$$

Where 'a' is the mean radiation absorption coefficient.

The boundary conditions for the problem are

$$\left. \begin{aligned} y^* = -\frac{d}{2} \quad u^* = 0 \quad T^* = 0 \quad C^* = 0 \\ \text{and} \\ y^* = \frac{d}{2} \quad u^* = 0, T^* = T_0 \cos \omega^* t^* \quad C^* = C_0 \cos \omega^* t^* \end{aligned} \right\} \quad (6)$$

where T_0 is the mean temperature, C_0 is the mean concentration, ω^* is the frequency of oscillations,

Introducing the following non-dimensional quantities

$$\eta = \frac{y^*}{d}, u = \frac{u^*}{U}, T = \frac{T^*}{T_0}, \omega = \frac{\omega^* d}{U}, t = \frac{t^* U}{d}, p = \frac{p^*}{\rho U^2} \quad Re = \frac{U d}{\nu} = \text{Reynolds number},$$

$$G_r = \frac{g\beta T_0 d^2}{U \nu^2} = \text{Grashof number}, G_m = \frac{g\beta_c T_0 d^2}{U \nu^2} = \text{Modified Grashof number}, M = B_0 d \sqrt{\frac{\sigma}{\mu}} = \text{Hartmann number},$$

$$N = \frac{2ad}{\sqrt{K}} = \text{Radiation parameter}, Pe = \frac{\rho C_p d U}{k} = \text{Peclet number}, Sc = \frac{\nu}{d} = \text{Schmidt number}, K_0 = \frac{K_0^* U}{d\nu} =$$

Viscoelastic parameter, $\chi = \frac{D_1 d^2}{\nu} = \text{Chemical reaction parameter}, K = \frac{k^*}{d^2} = \text{Permeability of the Porous medium}$

in equation (2) to (4) and using equation (5), the governing equation reduced to following non dimensional form

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial \eta} = -\frac{\partial P}{\partial \eta} + \frac{1}{Re} \frac{\partial^2 u}{\partial \eta^2} + \alpha \frac{\partial^3 u}{\partial t \partial \eta^2} - \frac{M^2}{Re} u + \frac{G_r}{Re} T + \frac{G_m}{Re} C - \nu \frac{u}{K} \quad (7)$$

$$\frac{\partial T}{\partial t} + \frac{\partial T}{\partial \eta} = \frac{1}{Pe} \frac{\partial^2 T}{\partial \eta^2} - \frac{N^2}{Pe} T \quad (8)$$

$$\frac{\partial C}{\partial t} + \frac{\partial C}{\partial \eta} = \frac{1}{Re Sc} \frac{\partial^2 C}{\partial \eta^2} - \frac{\chi}{Re} C \quad (9)$$

The boundary conditions in non-dimensional form become

$$\left. \begin{aligned} \eta = -\frac{1}{2} \quad u = 0, \quad T = 0 \quad C = 0 \\ \text{and} \\ \eta = \frac{1}{2} \quad u = 0, \quad T = \cos \omega t \quad C = \cos \omega t \end{aligned} \right\} \quad (10)$$

III. METHOD OF SOLUTION

In order to solve the equation (7), (8), (9) subjected to the boundary conditions (10) we assume the solution of the form,

$$-\frac{\partial P}{\partial x} = A e^{i\omega t}, u(\eta, t) = u_0(\eta) e^{i\omega t}, \theta(\eta, t) = \theta_0(\eta) e^{i\omega t}, \phi(\eta, t) = \phi_0(\eta) e^{i\omega t} \quad (11)$$

where A is the amplitude of pressure gradient and ω is the frequency of oscillations.

Substituting (11) in equations (7) to (9), we have

$$l_1^2 \frac{\partial^2 u_0}{\partial \eta^2} - Re \frac{\partial u_0}{\partial \eta} - l_2^2 u_0 = AR_e - G_r \theta_0 - G_m \phi_0 \quad (12)$$

$$\frac{\partial^2 \theta_0}{\partial \eta^2} - Pe \left(\frac{\partial \theta_0}{\partial \eta} \right) - m^2 \theta_0 = 0 \quad (13)$$

$$\frac{\partial^2 \phi_0}{\partial \eta^2} - Re Sc \frac{\partial \phi_0}{\partial \eta} - n^2 \phi_0 \quad (14)$$

The corresponding boundary conditions are

$$\left. \begin{aligned} \eta = -\frac{1}{2}, \quad u_0 = 0, \quad \theta_0 = 0, \quad \phi_0 = 0 \\ \text{and} \\ \eta = \frac{1}{2}, \quad u_0 = 0, \quad \theta_0 = 1, \quad \phi_0 = 1, \end{aligned} \right\} \quad (15)$$

Solving the equations (12), (13), (14) subjected to the boundary conditions (15) the expression for the velocity, temperature and concentration field are

$$u = \left(C_5 e^{A_5 \eta} + C_6 e^{A_6 \eta} + \frac{A_5 R_e}{12} - B_5 e^{A_1 \eta} + B_6 e^{A_2 \eta} - B_7 e^{A_3 \eta} + B_8 e^{A_4 \eta} \right) e^{i\omega t} \quad (16)$$

$$\theta = \left(\frac{e^{\frac{A_1 \eta - A_2}{2}} - e^{\frac{A_2 \eta - A_1}{2}}}{e^{\frac{A_1 - A_2}{2}} - e^{\frac{A_2 - A_1}{2}}} \right) e^{i\omega t} \quad (17)$$

$$\phi = \left(\frac{e^{\frac{A_3 \eta - A_4}{2}} - e^{\frac{A_4 \eta - A_3}{2}}}{e^{\frac{A_3 - A_4}{2}} - e^{\frac{A_4 - A_3}{2}}} \right) e^{i\omega t} \quad (18)$$

where all the constants used are given in the appendix.

IV. SKIN FRICTION, RATE OF HEAT AND MASS TRANSFER COEFFICIENTS:

The shear stress at the left plate of the channel in term of amplitude and phase angle is given by

$$\tau = \mu \left(\frac{\partial u}{\partial y} \right)_{y=-\frac{1}{2}} = |D| \cos(t + \upsilon) \quad (19)$$

where $D_r + iD_i = A_5 C_5 e^{-\frac{A_5}{2}} + A_6 C_6 e^{-\frac{A_6}{2}} - A_1 B_5 e^{-\frac{A_1}{2}} + A_2 B_6 e^{-\frac{A_2}{2}} - A_3 B_7 e^{-\frac{A_3}{2}} + A_4 B_8 e^{-\frac{A_4}{2}}$

where the amplitude $|D| = \sqrt{D_r^2 + D_i^2}$ and the phase angle $\upsilon = \tan^{-1} \frac{u_i}{u_r}$

The rate of heat transfer i.e. Nusselt number across the channel's left plate in term of amplitude and phase angle is given by

$$N_u = \left(\frac{\partial \theta}{\partial y} \right)_{y=-\frac{1}{2}} = |H| \cos(t + \psi) \quad (20)$$

where $H_r + iH_i = \left(\frac{A_1 - A_2}{e^{A_1 - e^{A_2}}} \right) e^{i\omega t}$

The amplitude $H = \sqrt{H_r^2 + H_i^2}$ and the phase angle $\psi = \tan^{-1} \frac{H_i}{H_r}$

The rate of mass transfer i.e. Sherwood number across the channel's left plate in term of amplitude and phase is given by

$$S_h = \left(\frac{\partial \phi}{\partial y} \right)_{y=-\frac{1}{2}} = |F| \cos(t + \Psi) \quad (21)$$

where $F_r + iF_i = \left(\frac{A_3 - A_4}{e^{A_3 - e^{A_4}}} \right) e^{i\omega t}$

the amplitude $F = \sqrt{F_r^2 + F_i^2}$ and the phase angle $\Psi = \tan^{-1} \frac{F_i}{F_r}$

V. RESULTS AND DISCUSSION

To study the effect of non-dimensional parameters on the flow process we have carried out the calculation for velocity, temperature, concentration profile.

The effect of the various parameters on the velocity profile is presented in figures (2-4). It is observed from the figs. that flow get accelerated with the increase of Reynolds number R_e , the amplitude of pressure gradient A , Peclet number P_e , Grashof number G_r , Modified Grashof number G_m , radiation parameter N and decelerated with the increase of Hartmann number M , Schmidt number S_c , viscoelastic parameter K_0 , chemical reaction parameter χ , Permeability of the Porous medium K and frequency of oscillation ω .

The amplitude of skin friction $|D|$ at the plate $\left(\eta = -\frac{1}{2} \right)$ is presented in figures (5-7). It is observed from these figures that amplitude of skin friction increase with the increase of the Reynolds number R_e , the amplitude of pressure gradient A , Peclet number P_e , Grashof number G_r , Modified Grashof number G_m and radiation parameter N and decrease with all other parameter.

The amplitude of heat transfer $|F|$ at the plate $\left(\eta = -\frac{1}{2} \right)$ is depicted in figure 8. It is observed that it decreases with the increase of Peclet number P_e and radiation parameter N .

The amplitude of mass transfer $|H|$ at the plate $\left(\eta = -\frac{1}{2} \right)$ is presented in figure 9. It is clear from this figure that it diminishes with the increase of Reynolds number R_e , chemical reaction parameter χ and Schmidt number S_c .

VI. PHYSICAL INTERPRETATIONS OF THE RESULTS

Physically if $(G_r > 0)$, it means cooling of the plate (or heating of the fluid) i.e. in free convection current which transfer heat away from the plate into the boundary layer region, therefore increasing value of Grashof number G_r accelerates the flow. The presence of magnetic field in electrically conducting fluid introduces a Lorentz force which acts against the flow and act as resistive force which slow down the flow. The S_c Schmidt number characterises the relative effectiveness of momentum and mass transport by diffusion, for higher value of S_c Schmidt number the species diffusivity rate exceed the momentum diffusivity which diminish

the concentration in the boundary layer. Further as the viscoelastic K_0 parameter increases, the hydrodynamics boundary layer adheres strongly to the surface which in turn retards the fluid motion.

VII. CONCLUSION

In the present paper flow of chemically reacting and radiating MHD oscillatory viscoelastic fluid through porous channel is studied. The closed form solution of the governing equation under the prescribed boundary condition is obtained. The conclusion of the study is as follows

1. The velocity decreases significantly with the increase of Permeability of the Porous medium, viscoelastic parameter, Hartmann number, chemical reaction and frequency of oscillation while increase with radiation parameter.
2. The amplitude of skin friction decreases with the Permeability of the Porous medium, viscoelastic parameter, Hartmann number and frequency of oscillation while increase with radiation parameter.
3. The amplitude of mass transfer decreases with Reynolds number, Schmidt number and chemical reaction parameter.

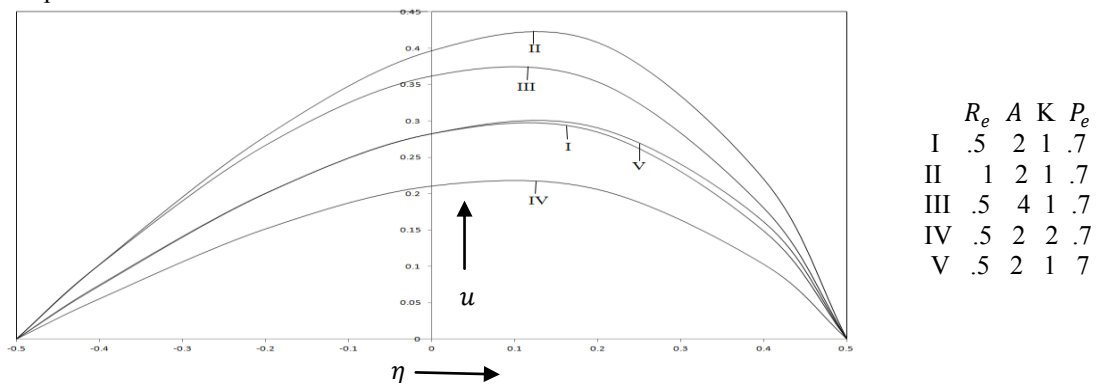


Figure 2 Velocity profile for $G_m = 5$, $M = 2$, $N = 2$, $\chi = 0.2$, $S_c = .22$, $K_0 = 0.05$, $\omega = 5$ and $t = 0$.

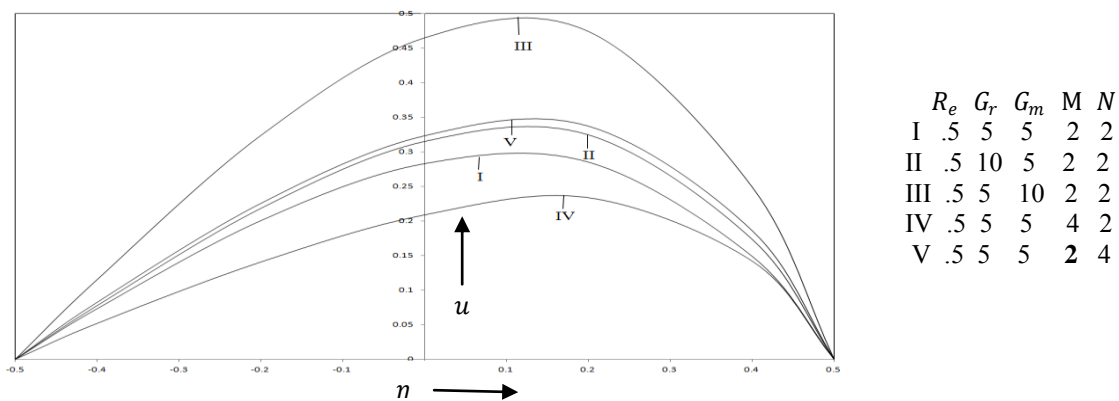


Figure 3 Velocity profile for $A = 2$, $k = 1$, $P_e = .7$, $\chi = 0.2$, $S_c = .22$, $K_0 = 0.05$, $\omega = 5$ and $t = 0$.

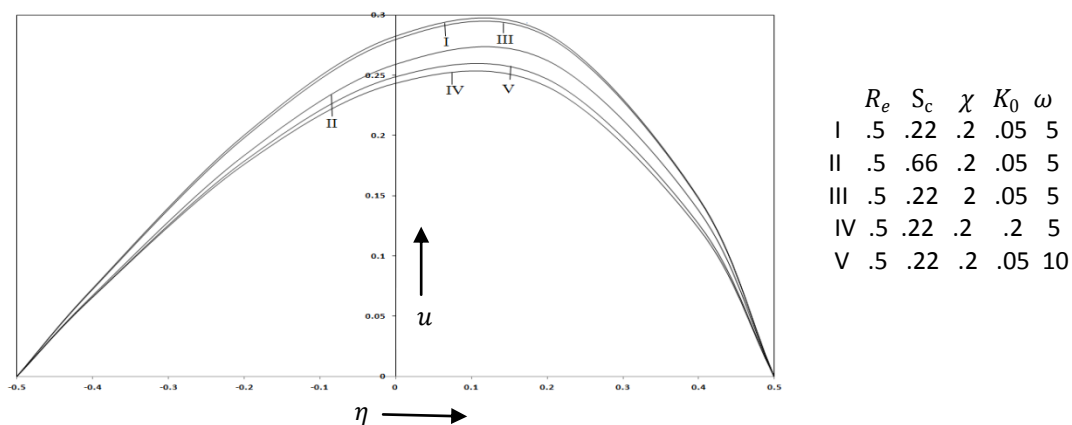


Figure 4 Velocity profile for $A = 2$, $k = 1$, $P_e = .7$, $G_r = 5$, $G_m = 5$, $M = 2$, $N = 2$, $\omega = 5$ and $t = 0$.

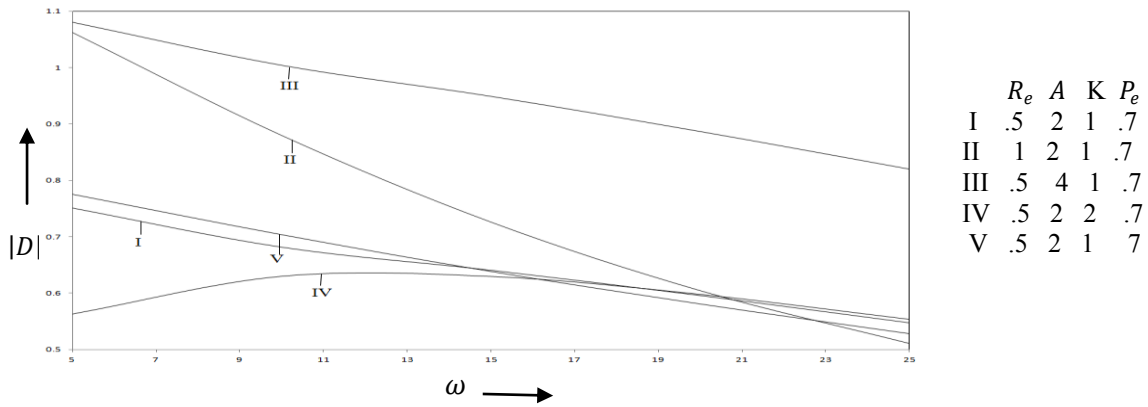


Figure 5 Amplitude $|D|$ of Skin friction for $G_m = 5$, $M = 2$, $N = 2$, $\chi = 0.2$, $S_c = .22$, $K_0 = 0.05$, $\omega = 5$ and $t = 0$.

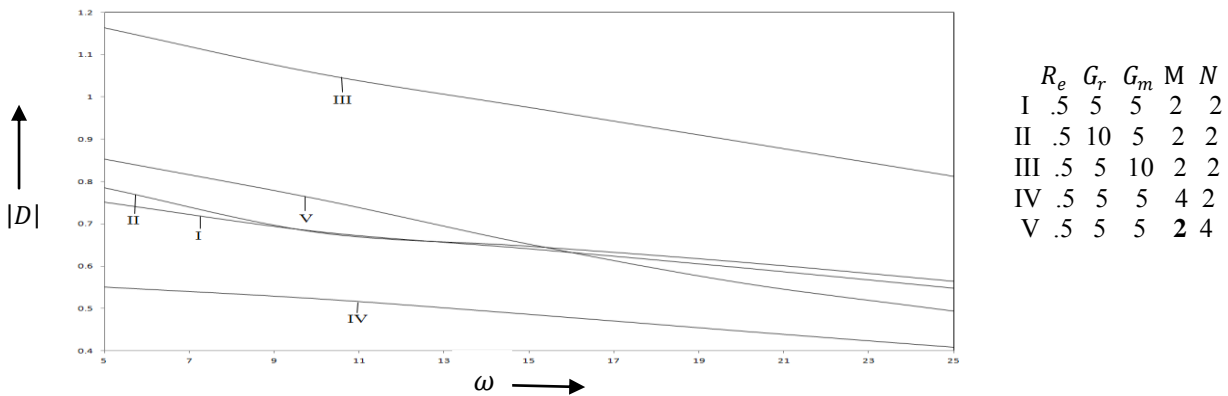


Figure 6 Amplitude $|D|$ of Skin friction $A = 2$, $k = 1$, $P_e = .7$, $\chi = 0.2$, $S_c = .22$, $K_0 = 0.05$, $t = 0$.

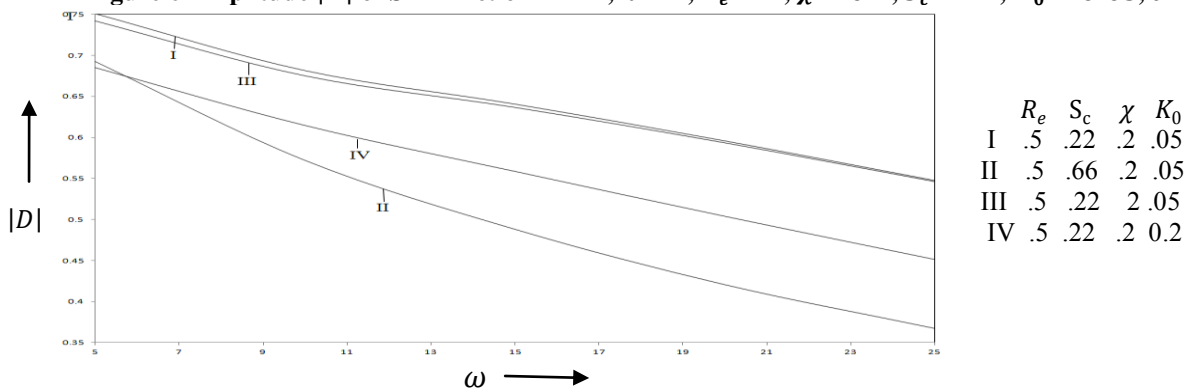


Figure 7 Amplitude $|D|$ of Skin friction for $A = 2$, $k = 1$, $P_e = .7$, $G_r = 5$, $G_m = 5$, $M = 2$, $N = 2$, $\omega = 5$, $t = 0$.

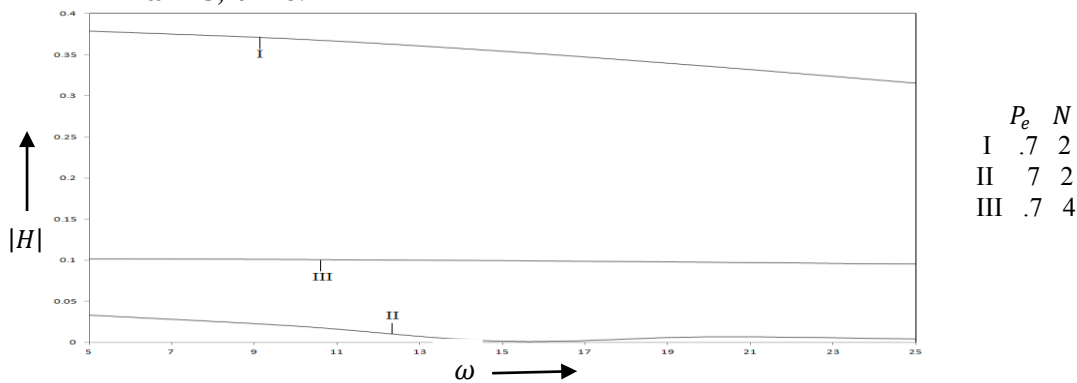


Figure 8 Amplitude $|H|$ of Heat transfer

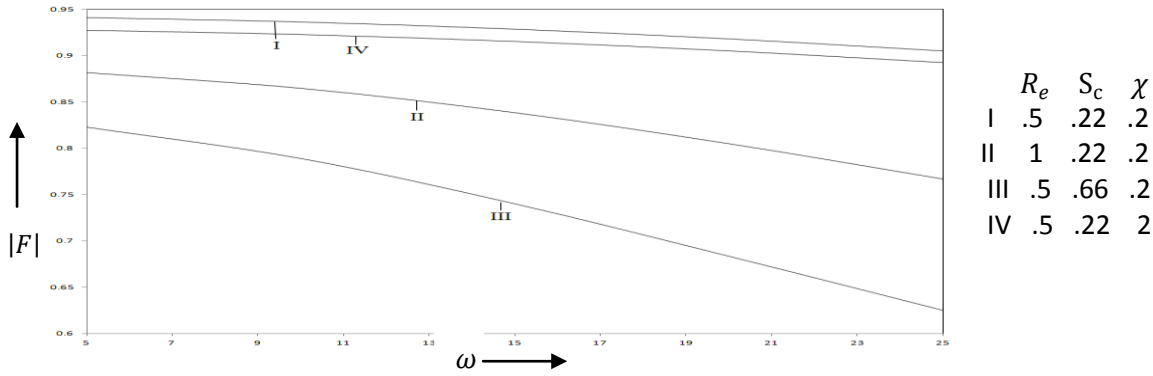


Figure 9 Amplitude $|F|$ of mass transfer

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APPENDIX

$$l_1 = \sqrt{1 + i\omega\alpha}, \quad l_2 = \sqrt{M^2 + k^{-1} + i\omega R_e}, \quad A_1 = \frac{p_e + \sqrt{(p_e)^2 + 4m^2}}{2}, \quad A_2 = \frac{p_e - \sqrt{(p_e)^2 + 4m^2}}{2},$$

$$A_3 = \frac{ReSc + \sqrt{(ReSc)^2 + 4n^2}}{2}, \quad A_4 = \frac{ReSc - \sqrt{(ReSc)^2 + 4n^2}}{2}, \quad A_5 = \frac{Re + \sqrt{(Re)^2 + 4l_1^2 l_2^2}}{2l_1^2}, \quad A_6 = \frac{Re - \sqrt{(Re)^2 + 4l_1^2 l_2^2}}{2l_1^2},$$

$$B_1 = \frac{G_r e^{-\frac{A_2}{2}}}{2 \sinh \frac{A_1 - A_2}{2}}, \quad B_2 = \frac{G_r e^{-\frac{A_1}{2}}}{2 \sinh \frac{A_1 - A_2}{2}}, \quad B_3 = \frac{G_m e^{-\frac{A_4}{2}}}{2 \sinh \frac{A_3 - A_4}{2}}, \quad B_4 = \frac{G_m e^{-\frac{A_3}{2}}}{2 \sinh \frac{A_3 - A_4}{2}},$$

$$B_5 = \frac{B_1}{l_1^2 A_1^2 - Re A_1 - l_2^2}, \quad B_6 = \frac{B_2}{l_1^2 A_2^2 - Re A_2 - l_2^2}, \quad B_7 = \frac{B_3}{l_1^2 A_3^2 - Re A_3 - l_2^2}, \quad B_8 = \frac{B_4}{l_1^2 A_4^2 - Re A_4 - l_2^2},$$

$$D_1 = \frac{AR_e}{l_2^2} - B_5 e^{-\frac{A_1}{2}} + B_6 e^{-\frac{A_1}{2}} - B_7 e^{-\frac{A_3}{2}} + B_8 e^{-\frac{A_4}{2}}$$

$$D_2 = \frac{AR_e}{l_2^2} - B_5 e^{\frac{A_1}{2}} + B_6 e^{\frac{A_2}{2}} - B_7 e^{\frac{A_3}{2}} + B_8 e^{\frac{A_4}{2}}$$