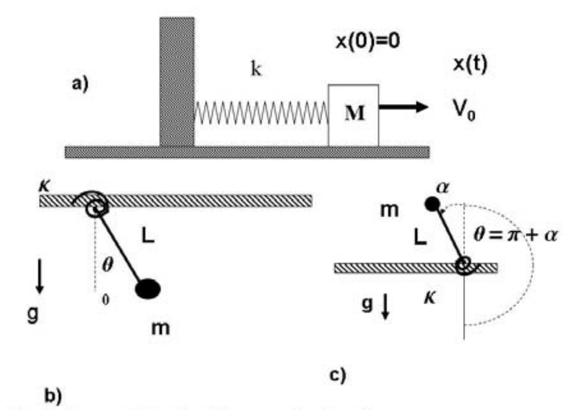
Vibration, Normal Modes, Natural Frequencies, Instability

Vibration, Instability

An important class of problems in dynamics concerns the free vibrations of systems. (The concept of free vibrations is important; this means that although an outside agent may have participated in causing an initial displacement or velocity–or both– of the system, the outside agent plays no further role, and the subsequent motion depends only upon the inherent properties of the system. This is in contrast to "forced" motion in which the system is continually driven by an external force.) We shall consider only undamped systems for which the total energy is conserved and for which the frequencies of oscillation are real. This forms the basis of the approach to more complex studies for forced motion of damped systems. We saw in Lecture 13, that the free vibration of a mass-spring system could be described as an oscillatory interchange between the kinetic and potential energy, and that we could determine the natural frequency of oscillation by equating the maximum value of these two quantities. (The natural frequency is the frequency at which the system will oscillate unaffected by outside forces. When we consider the oscillation of a pendulum, the gravitational force is considered to be an inherent part of the system.) The general behavior of a mass-spring system can be extended to elastic structures and systems experiencing gravitational forces, such as a pendulum. These systems can be combined to produce complex results, even for one-degree of freedom systems.

We begin our discussion with the solution of a simple mass-spring system, recognizing that this is a model for more complex systems as well.



Single degree of freedom "mass-spring" systems

In the figure, a) depicts the simple mass spring system: a mass M, sliding on a frictionless plane, restrained by a spring of spring constant k such that a force $F(x) = -kx$ opposes the displacement x. (In a particular problem, the linear dependence of the force on x may be an approximation for small x.) In order to get a solution, the initial displacement and initial velocity must be specified. Common formulations are: $x(0) = 0$, and $\frac{dx}{dt}(0) = V_0$ (The mass responds to an initial impulse.); or $x(0) = X_0$ and $\frac{dx}{dt}(0) = 0$ (The mass is given an initial displacement.). The general formulation is some combination of these initial conditions. From Newton's law, we obtain the governing differential equation

$$m\frac{d^2x}{dt^2} = -kx \tag{1}$$

with $x(0) = X_0$, and $\frac{dx}{dt}(0) = V_0$.

The solution is of the general form, $x(t) = Re(Ae^{i\omega t})$, where, at this point in the analysis, ***both A and ω are unknown***. That is, we assume a solution in which both A and ω are unknown, and later when the solution is found and boundary conditions are considered, we will end up taking the real part of the expression. Depending upon whether A is purely real, purely imaginary, or some combination, we will in general get oscillatory behavior involving $\sin' s$ and $cos's$ since $e^{i\omega t} = \cos\omega t + i\sin\omega t$. For a system without damping, $\omega^2 = Real(K)$ (K being some combination of system parameters.), so that $\omega = \pm\sqrt{K}$. For undamped systems, these two $\pm$ values of ω, are redundant; only one need be taken.

To solve the differential equation, equation(1) is rewritten

$$m\frac{d^2x}{dt^2} + kx = 0. \tag{2}$$

With the assumed form of solution, this becomes

$$-\omega^2 mx(t) + kx(t) = 0 = x(t) * (-\omega^2 m + k) \tag{3}$$

Since $x = x(t)$, for a valid solution, we require $\omega = \sqrt{\frac{k}{m}}$.

This approach works as well for systems b) and c). These two systems are pendulums restrained by torsion springs, which for small angles (θ or α) produce a restoring torque proportional to the angular departure from equilibrium. Consider system b). Its equilibrium position is $\theta = 0$. The restoring torque from the spring is $T_s = -\kappa\theta$. The restoring torque from gravity is $T_g = -mgLsin\theta$ which for small angles becomes $T_g = -mgL\theta$. Writing Newton's law in a form appropriate for pendular motion, we obtain

$$mL\frac{d^2\theta}{dt^2} = \frac{1}{L}(T_g + T_s) \tag{4}$$

We assume a form of solution, $\theta(t) = Ae^{(i\omega t)}$, and rewrite the equation as before, moving all terms to the left-hand side.

$$\left(-mL\omega^2 + \frac{1}{L}(mgL + \kappa)\right)\theta(t) = 0. \tag{5}$$

Therefore for a solution we require

$$\omega = \sqrt{\frac{mgL + \kappa}{mL^2}} \tag{6}$$

Examining this result, we see that the combination of the spring and gravity acts to increase the natural frequency of the oscillation. Also if there is no spring, $\kappa = 0$, and the result becomes just the frequency of a pendulum $\omega = \sqrt{\frac{g}{L}}$.

System c) is perhaps a bit more interesting. In this case, we use the small angle α. We take the equilibrium position of the spring to be $\alpha = 0$ so that the restoring torque due to the spring is again $T_s = -\kappa\alpha$. But now in this case, we are expanding the gravitational potential about the point $\alpha = 0$. Since this is an unstable equilibrium point, this gives the restoring (–it doesn't restore, it keeps going!–) torque due to gravity as $T_g = mgLsin\alpha$ or for small α, $T_g = mgL\alpha$.

Writing the governing equation for this case, we obtain

$$\left(-mL\omega^2 + \frac{1}{L}(-mgL + \kappa)\right)\alpha(t) = 0. \tag{7}$$

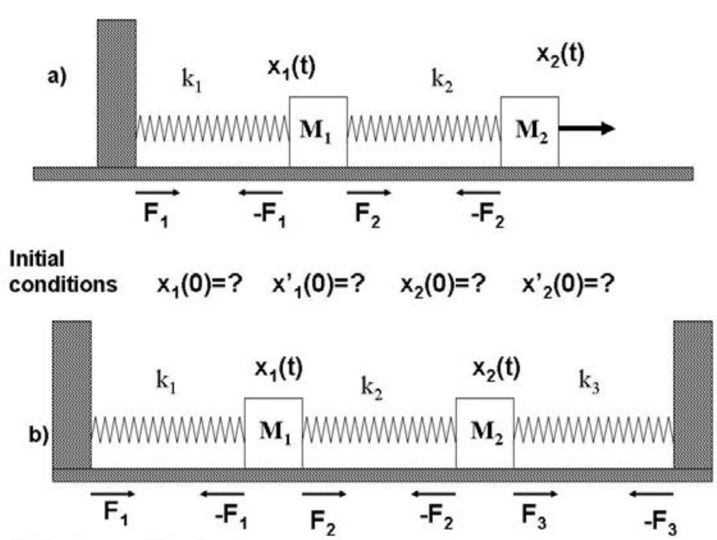
Therefore for a solution we require

$$\omega = \sqrt{\frac{-mgL + \kappa}{mL^2}} \tag{8}$$

In this case, there is a critical value of κ for which $\omega = 0$. On either side of this point, we have $\omega = \sqrt{\frac{\kappa - mgL}{mL^2}}$, which gives $\omega \longrightarrow real$ for $\kappa > mgL$, and $\omega \longrightarrow imag$ for $\kappa < mgL$. Since we have assumed $\alpha(t) = e^{i\omega t}$, a real ω will produce oscillatory motion; an imaginary ω will produce exponentially diverging, or unstable, motions. We say that the pendulum for κ less than the critical value, $\kappa = mgL$, is unstable.

Vibration of Multi-Degree of Freedom Systems

We begin our treatment of systems with multiple degrees of freedom, by considering a two degree of freedom system. This system contains the essential features of multi-degree of freedom systems. Consider the ***two*** two-mass, two-spring systems shown in the figure.



Multi-degree of freedom systems

In this case, there are two independent variables, $x_1(t)$ and $x_2(t)$; their motion is not independent, but is coupled by their attachments to the springs k_1, k_2 and for system b), k_3. The sketch shows the forces F_i acting on the masses as a result of the extension of the spring; these of are equal and opposite at the ends of the springs. We consider both system a) and b). System b) is actually the simpler of the two systems because of its inherent symmetry.

The governing equations can be written as

for system a)

$$m_1 \frac{d^2 x_1}{dt^2} = -k_1 x_1 + k_2(x_2 - x_1) \tag{9}$$

$$m_2 \frac{d^2 x_2}{dt^2} = -k_2(x_2 - x_1) \tag{10}$$

for system b)

$$m_1 \frac{d^2 x_1}{dt^2} = -k_1 x_1 + k_2(x_2 - x_1) \tag{11}$$

$$m_2 \frac{d^2 x_2}{dt^2} = -k_2(x_2 - x_1) - k_3 x_2 \tag{12}$$

In both cases, as before, we assume a solution of the form $x_1(t) = X_1 e^{i\omega t}$ and $x_2(t) = X_2 e^{i\omega t}$. However, as we will see, in this case, we will obtain two possible values for ω^2; both will be real; we will take only the positive value of ω itself. These will be the two vibration modes of this two degree of freedom system. These results extend to N ω^2's for an N degree of freedom system. Again, we will take only the positive value of ω.

Consider first system b). With the assumed form of solution, and rewriting all terms on the left-hand side, we obtain

$$-\omega^2 m_1 X_1 + k_1 X_1 - k_2(X_2 - X_1) = 0 \tag{13}$$

$$-\omega^2 m_2 X_2 + k_2(X_2 - X_1) + k_3 X_2 = 0 \tag{14}$$

This equation can be written in matric form as

$$\left(\begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{pmatrix} - \begin{pmatrix} m_1\omega^2 & 0 \\ 0 & m_2\omega^2 \end{pmatrix} \right) \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{15}$$

or

$$\begin{pmatrix} k_1 + k_2 - m_1\omega^2 & -k_2 \\ -k_2 & k_2 + k_3 - m_2\omega^2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{16}$$

This equation makes a very powerful statement. Since the right-hand side of both equations is zero, a condition for a solution is that the determinant of the matrix equals zero. This will give an algebraic equation with two solutions for ω: ω_1 and ω_2. These are the "natural" frequencies of the two degree of freedom system. In the general case, they are not equal; and both x_1 and x_2 participate in the oscillation at each frequency ω_i. Also, as in the single degree of freedom system, the actual values of $x_1(t)$ and $x_2(t)$ are determined by initial conditions; in this case 4 initial conditions are required: $x_1(0)$, $x_2(0)$, $\frac{dx_1}{dt}(0)$,and $\frac{dx_2}{dt}(0)$. The actual values of X_1 and X_2 are of less interest than the relationships between them and the structure of the problem.

If the two masses are equal, a particularly simple form of a more general result follows from equation(16). We consider this as an introduction to the more general case. The more general case will be considered shortly.

$$\begin{pmatrix} k_1/m + k_2/m - \omega^2 & -k_2/m \\ -k_2/m & k_2/m + k_3/m - \omega^2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{17}$$

We determine the two values of ω_i (ω_1 and ω_2) by setting the determinant equal to zero. We then substitute each value of ω_i in turn into the matrix equation and determine for each ω_i the coefficients X_{1i} and X_{2i}; only their ratio can be determined. We write the coefficients X_{1i} and X_{2i} as vectors, $\vec{X}_1 = (X_{11}, X_{21})$ and $\vec{X}_2 = (X_{12}, X_{22})$, where the subscript 1 refers to the mode associated with ω_1, and 2 refers to the mode associate with ω_2.

It is a remarkable property of the solution to the governing equations that these vectors are orthogonal: the dot product of $\vec{X}_1 \cdot \vec{X}_2 = 0$ (We will follow with an example to amplify and clarify this.)

Consider the simplest case of system b) with both masses equal to m and all springs of stiffness k. In this case we have

$$\begin{pmatrix} 2k/m - \omega^2 & -k/m \\ -k/m & 2k/m - \omega^2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{18}$$

Setting the determinant equal to zero gives two solutions for ω: $\omega_1 = \sqrt{k/m}$ and $\omega_2 = \sqrt{3k/m}$. The components of the x_1 and x_2 motion are: $\vec{X}_1 = (1, 1)$ and $\vec{X}_2 = (-1, 1)$. This simple example gives great physical insight to the more general problem. The natural frequency is ω_i; the components $\vec{X}_i = (X_{i1}, X_{i2})$ are called "normal modes".

Normal Modes of Multi-Degree of Freedom Systems

Examining the first "normal mode", we see an oscillation in which $\vec{X}_1 = (1, 1)$ occurs at an oscillation frequency $\omega_1 = \sqrt{k/m}$. Since $\vec{X}_1 = (1, 1)$, the central spring does not deform, and the two masses oscillate, each on a single spring, thus giving a frequency $\omega = \sqrt{k/m}$.

The second "normal mode" has a frequency $\omega = \sqrt{3k/m}$, with $\vec{X}2 = (-1, 1)$; thus the masses move in opposite directions, and the frequency of oscillation is increased. It can be seen by inspection that the vector $\vec{X}_1$ and $\vec{X}_2$ are orthogonal (their dot product is zero.)

If such a system was at rest, and an initial impulse was given to ***one*** of the masses, ***both*** modes would be excited and a free oscillation would occur with each "mode" oscillating at "its" natural frequency.

The equation for general values of k_1, k_2 and k_3 can be written

$$\left(\begin{pmatrix} k_1/m + k_2/m & -k_2/m \\ -k_2/m & k_2/m + k_3/m \end{pmatrix} - \omega^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{19}$$

Characteristic Value Problem

This problem is called a characteristic-value or eigenvalue problem. Formulated in matrix notation it can be written

$$\left(\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{20}$$

The requirements on the simplest form of the characteristic value problem are that the matrix [A] is symmetric (This will always be true for combinations of masses and springs.), and that the characteristic value λ multiplies an identity matrix. (An identity matrix has all 1's on the diagonal and 0's off the diagonal; this will be true if all the masses are equal; if not, a more general form must be used yielding analogous results. This more general from will be considered shortly.)

For the form of the governing equation shown in equation(20), the characteristic value $\lambda = \omega^2$. The general solution to this problem will yield a set of solutions for λ equal to the size of the matrix :(i.e a 4×4 matrix will result in 4 λ's). For each λ_i, a vector $\vec{X}_i$ will be obtained. For this form of the characteristic value problem, the dot product between any two of these vectors is zero.

$$(X_{i,1}, X_{i,2}).(X_{j,1}, X_{j,2}) = 0 \tag{21}$$

for $i \neq j$. These are the ***normal modes*** of the system, and the ω's are the ***natural frequencies***. Any numerical matrix method–such as MATLAB– will yield both the λ_i's (called the eigenvalues) and the X_i's, called the eigenvectors for a particular matrix [A]. A similar result is obtained for the modes of vibration of a continuous system such as a beam. The displacement of the various mode of vibration of a uniform beam are orthogonal.

The general solution for the motion of the masses is then given by an expansion in the ***normal modes***, $\vec{X}_i$,

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = A_1 e^{i\omega_1 t}\begin{pmatrix} X_{11} \\ X_{12} \end{pmatrix} + A_2 e^{i\omega_2 t}\begin{pmatrix} X_{21} \\ X_{22} \end{pmatrix}, \tag{22}$$

where $\vec{X}_1 = (X_{11}, X_{21})$ and $\vec{X}_2 = (X_{12}, X_{22})$ are the eigenvectors or normal modes from the solution of the characteristic-value problem, obtained by hand or numerically, and ω_i is the natural frequency of that mode. Again, it is a remarkable and extremely useful property that the dot product of $\vec{X}_i$ and $\vec{X}_j$ is zero unless $i = j$. Good form would suggest that we normalize each X_i so that the magnitude of $X_i \cdot X_i$ equals 1, but we usually don't and therefore need to define $C_i = \vec{X}_i \cdot \vec{X}_i$, the vector-magnitude-squared, for later use.

Expansion in Normal Modes; Satisfaction of Initial Conditions

The general form of solution is given by equation (22). All that remains is to determine the coefficients A_1 and A_2. This is done by satisfying the initial conditions on displacement and velocity. In the general case, since $e^{i\omega t} = 1$ at $t = 0$, the initial displacement can be written

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = A_1 \begin{pmatrix} X_{11} \\ X_{12} \end{pmatrix} + A_2 \begin{pmatrix} X_{21} \\ X_{22} \end{pmatrix}. \tag{23}$$

and for an initial velocity, since $i\omega e^{i\omega t} = i\omega$ at $t = 0$, the initial velocity can be written

$$\begin{pmatrix} v_1(0) \\ v_2(0) \end{pmatrix} = i\omega A_1 \begin{pmatrix} X_{11} \\ X_{12} \end{pmatrix} + i\omega A_2 \begin{pmatrix} X_{21} \\ X_{22} \end{pmatrix}. \tag{24}$$

It should be noted that in general A is complex; the real part relates to the initial displacement; the imaginary part to the initial velocity. If we consider A_i to be real, we are automatically assuming no initial velocity.

Case 1: initial displacement non-zero; initial velocity zero

We first consider the case of an initial condition on the displacement, specifically $x_1(0) = x_{10}$ and $x_2(0) = x_{20}$, with $v_1(0) = 0$ and $v_2(0) = 0$. We define the initial-condition vector as $\vec{X_0} = (x_1(0), x_2(0))$. To complete the solution, we need to obtain the values of A_1 and A_2 from equation(23). This is done by taking the dot product of both sides of equation(23) with the first mode $\vec{X_1}$. Finally we get our payoff for all our hard work. Since the dot product of $\vec{X_1}$ with $\vec{X_2}$ is zero; the dot product of $\vec{X_1}$ with $\vec{X_1}$ is C_1; and the dot product of $\vec{X_0}$ with $\vec{X_1}$ is some G_1, we obtain

$$A_1 = \vec{X_0} \cdot \vec{X_1} / C_1 = \frac{G_1}{C_1} \tag{25}$$

and taking the dot product of $\vec{X_0}$ with $\vec{X_2}$ (which equals some G_2), we obtain

$$A_2 = \vec{X_0} \cdot \vec{X_2} / C_2 = \frac{G_2}{C_2} \tag{26}$$

With A_1 and A_2 determined, we have a complete solution to the problem.

Case 2: initial displacement zero; initial velocity non-zero

We now consider the case of an initial condition on the velocity, specifically $\dot{x}_1(0) = v_{10}$ and $\dot{x}_2(0) = v_{20}$, with $x_1(0) = 0$ and $x_2(0) = 0$. We define the initial-condition vector as $\vec{V_0} = (v_{10}, v_{20})$. We use the coefficient B_i to define the solution for this case. The solution is again written as an expansion in normal modes oscillating at their natural frequency ω_i of amplitude B_i, which is unknown at this point.

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = B_1 e^{i\omega t} \begin{pmatrix} X_{11} \\ X_{12} \end{pmatrix} + B_2 e^{i\omega t} \begin{pmatrix} X_{21} \\ X_{22} \end{pmatrix}. \tag{27}$$

As previously noted, B is complex; the real part relates to the initial displacement; the imaginary part to the initial velocity. If B_i is imaginary, there is no initial displacement. The velocity at $t = 0$ is given by

$$\begin{pmatrix} v_1(0) \\ v_2(0) \end{pmatrix} = i\omega B_1 \begin{pmatrix} X_{11} \\ X_{12} \end{pmatrix} + i\omega B_2 \begin{pmatrix} X_{21} \\ X_{22} \end{pmatrix}. \tag{28}$$

To complete the solution, we need to obtain the values of B_1 and B_2 from equation(28). This is again done by taking the dot product of equation(28) with the first mode $\vec{X}_1$. Since the dot product of $\vec{X}_1$ with $\vec{X}_2$ is zero, and the dot product of $\vec{X}_1$ with $\vec{X}_1$ is C_1, we obtain

$$B_1 = -\frac{Q_1 i}{C_1 \omega} \tag{29}$$

where $Q_1 = \vec{V}_0 \cdot \vec{X}_1$ and taking the dot product of $\vec{X}_0$ with $\vec{X}_2$ we obtain

$$B_2 = -\frac{Q_2 i}{C_2 \omega} \tag{30}$$

where $Q_2 = \vec{V}_0 \cdot \vec{X}_2$ and $\vec{X}_2 \cdot \vec{X}_2 C_2 =$. The fact that B_i is purely imaginary confirms our earlier observation that real coefficients imply a non-zero initial displacement while purely imaginary coefficients imply a non-zero initial velocity. A purely imaginary B_i simply implies that the displacement has a $sin(\omega t)$ behavior in contrast to a $cos(\omega t)$ behavior, since the real part of $e^{i\omega t} = cos(\omega t)$, while the real part of $-ie^{i\omega t} = sin(\omega t)$. A solution for general initial conditions on $x_1(0)$, $x_2(0)$, $x_1\dot{(}0)$ and $x_2\dot{(}0)$ would be a linear combination of these solutions.

Solution for Unequal Masses

If the masses, m_i, are not equal we must use a more general form of the eigenvalue problem. Returning to Equation (19) for equal masses.

For the case of equal masses, from Equation (19), this can be written

$$\begin{pmatrix} k_1/m + k_2/m & -k_2/m \\ -k_2/m & k_2/m + k_3/m \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \omega^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{31}$$

For unequal masses, we rewrite the equation as

$$\begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \omega^2 \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \tag{32}$$

This equation is in the form of the generalized characteristic or eigenvalue problem. (See Hildebrand; Methods of Applied Mathematics.)

$$\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} - \lambda \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \right) \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{33}$$

where equation (20) has been rewritten as (33) in the extended formulation $([A] - \lambda[B])(X) = 0$. Following Hildebrand we note that both $[A]$ and $[B]$ are symmetric matrices. Moreover, [B] is a diagonal matrix. Following the solution of the generalized characteristic or eigenvalue problem, by numerical or other technique, the resulting orthogonality condition between the "normal" modes is modified as

$$(X_{i,1}, X_{i,2}) \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} X_{j,1} \\ X_{j,2} \end{pmatrix} = 0 \tag{34}$$

for $i \neq j$. Thus the vectors of displacement for the normal modes of vibration must be multiplied by the mass distribution to result in orthogonality. A similar result is obtained for the vibration of a continuous system, such as a beam with non-uniform mass distribution. The process of expanding the solution in term of normal modes goes through as before with the modification of the normality condition.

In more general configurations, $([A] - \lambda[B])(\mathbf{X}) = 0$, the matrix $[B]$ may not be diagonal, that is

$$[B] = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \tag{35}$$

As long as $[B]$ is symmetric, the orthogonalization goes through as

$$(X_{i,1}, X_{i,2}) \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} X_{j,1} \\ X_{j,2} \end{pmatrix} = 0 \tag{36}$$

Observations

The process we have outlined for finding the solution to the initial value problem to a multi-degree of freedom system, outlined from equation(20) on, works for system with degrees of freedom from 2 to 20,000 and beyond. This approach is of fundamental importance in analyzing vibrations in a wide variety of systems. The expansion in normal modes is also useful in more complex problems such as forced motions at frequencies other than ω_i.

References

[1] Hildebrand: Methods of Applied Mathematics; for a discussion of the characteristic value problem, matrices and vectors.