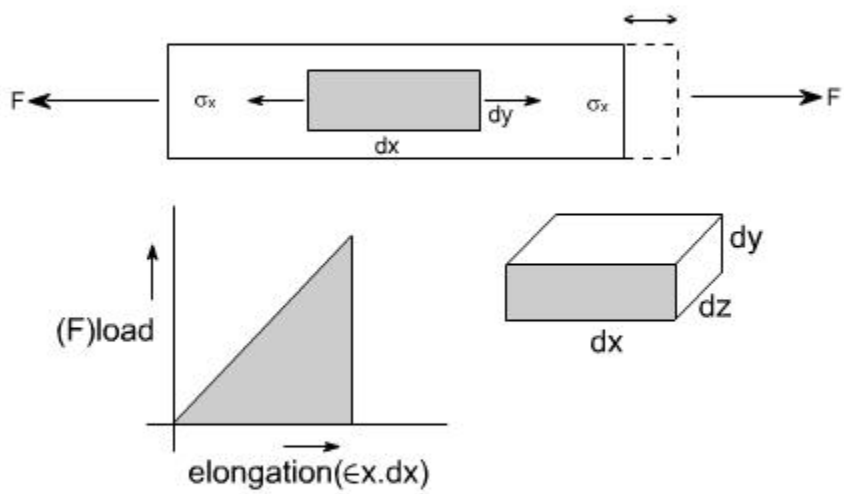
THEORIES OF ELASTIC FAILURE

While dealing with the design of structures or machine elements or any component of a particular machine the physical properties or chief characteristics of the constituent materials are usually found from the results of laboratory experiments in which the components are subject to the simple stress conditions. The most usual test is a simple tensile test in which the value of stress at yield or fracture is easily determined.

However, a machine part is generally subjected simultaneously to several different types of stresses whose actions are combined therefore, it is necessary to have some basis for determining the allowable working stresses so that failure may not occur. Thus, the function of the theories of elastic failure is to predict from the behavior of materials in a simple tensile test when elastic failure will occur under any conditions of applied stress.

A number of theories have been proposed for the brittle and ductile materials.

Strain Energy: The concept of strain energy is of fundamental importance in applied mechanics. The application of the load produces strain in the bar. The effect of these strains is to increase the energy level of the bar itself. Hence a new quantity called strain energy is defined as the energy absorbed by the bar during the loading process. This strain energy is defined as the work done by load provided no energy is added or subtracted in the form of heat. Some times strain energy is referred to as internal work to distinguish it from external work ‘W'. Consider a simple bar which is subjected to tensile force F, having a small element of dimensions dx, dy and dz.



The strain energy U is the area covered under the triangle

$$U = \frac{1}{2} F . \in_x . d_x$$

$$= \frac{1}{2} \sigma_x . dydz . dx \in_x$$

$$= \frac{1}{2} \sigma_x . \in_x . dxdydz$$

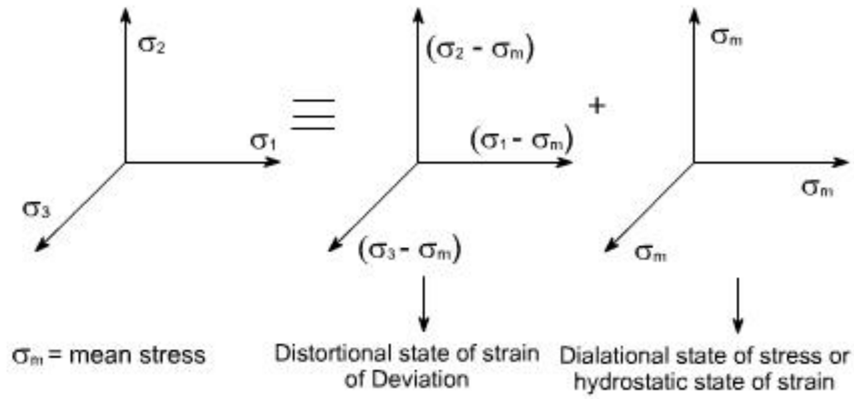
$$= \frac{1}{2} \sigma_x \left(\frac{\sigma_x}{E} \right) . dxdydz$$

$$\boxed{\frac{U}{volume} = \frac{1}{2} \frac{\sigma_x^{\,2}}{E}}$$

A three dimension state of stress respresented by σ_1, σ_2 and σ_3 may be throught of consisting of two distinct state of stresses i.e Distortional state of stress

Deviatoric state of stress and dilational state of stress

Hydrostatic state of stresses.



$$\sigma_m = \frac{(\sigma_1 + \sigma_2 + \sigma_3)}{3}$$

Thus, The energy which is stored within a material when the material is deformed is termed as a strain energy. The total strain energy U_r

$$U_T = U_d + U_H$$

U_d is the strain energy due to the Deviatoric state of stress and U_H is the strain energy due to the Hydrostatic state of stress. Futher, it may be noted that the hydrostatic state of stress results in change of volume whereas the deviatoric state of stress results in change of shape.

Different Theories of Failure : These are five different theories of failures which are generally used

(a) Maximum Principal stress theory (due to Rankine)

(b) Maximum shear stress theory (Guest - Tresca)

(c) Maximum Principal strain (Saint - venant) Theory

(d) Total strain energy per unit volume (Haigh) Theory

(e) Shear strain energy per unit volume Theory (Von – Mises & Hencky)

In all these theories we shall assume.

σ_{Yp} = stress at the yield point in the simple tensile test.

σ_1,□σ_2,□σ_3 – the three principal stresses in the three dimensional complex state of stress systems in order of magnitude.

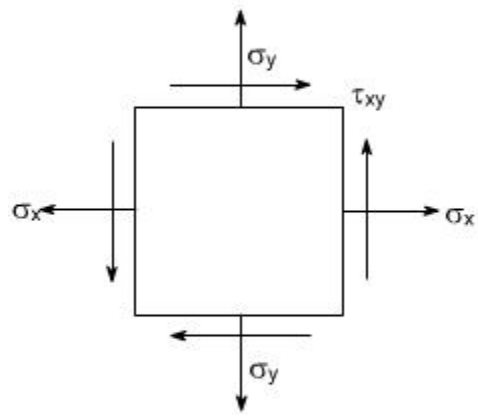
(a) Maximum Principal stress theory :

This theory assume that when the maximum principal stress in a complex stress system reaches the elastic limit stress in a simple tension, failure will occur.

Therefore the criterion for failure would be

$\sigma_1 = \sigma_{yp}$

For a two dimensional complex stress system σ_1 is expressed as



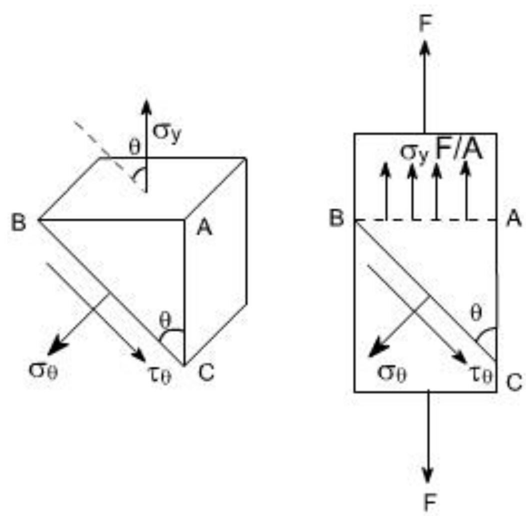
$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4.\tau_{xy}^2}$$
$$= \sigma_{yp}$$

Where σ_x, σ_y and τ_{xy} are the stresses in the any given complex stress system.

(b) Maximum shear stress theory:

This theory states that teh failure can be assumed to occur when the maximum shear stress in the complex stress system is equal to the value of maximum shear stress in simple tension.

The criterion for the failure may be established as given below :



For a simple tension case

$$\sigma_\theta = \sigma_y \sin^2\theta$$

$$\tau_\theta = \frac{1}{2}\sigma_y \sin 2\theta$$

$$\tau_\theta\big|_{max^m} = \frac{1}{2}\sigma_y \quad \text{or}$$

$$\tau_{max^m} = \frac{1}{2}\sigma_{yp}$$

whereas for the two dimentional complex stress system

$$\tau_{max^m} = \left(\frac{\sigma_1 - \sigma_2}{2}\right)$$

where σ_1 = maximum principle stress

σ_2 = minimum principal stress

$$\text{so} \quad \frac{\sigma_1 - \sigma_2}{2} = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2 xy}$$

$$\frac{\sigma_1 - \sigma_2}{2} = \frac{1}{2}\sigma_{yp} \Rightarrow \sigma_1 - \sigma_2 = \sigma_y$$

$$\Rightarrow \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2 xy} = \sigma_{yp}$$

becomes the criterion for the failure.

(c) Maximum Principal strain theory :

This Theory assumes that failure occurs when the maximum strain for a complex state of stress system becomes equals to the strain at yield point in the tensile test for the three dimensional complex state of stress system.

For a 3 - dimensional state of stress system the total strain energy U_t per unit volume in equal to the total work done by the system and given by the equation

$$U_t = 1/2\sigma_1 \in_1 + 1/2\sigma_2 \in_2 + 1/2\sigma_3 \in_3$$

substituting the values of $\in_1, \in_2$ and $\in_3$

$$\in_1 = \frac{1}{E}\left[\sigma_1 - \gamma(\sigma_2 + \sigma_3)\right]$$

$$\in_2 = \frac{1}{E}\left[\sigma_2 - \gamma(\sigma_1 + \sigma_3)\right]$$

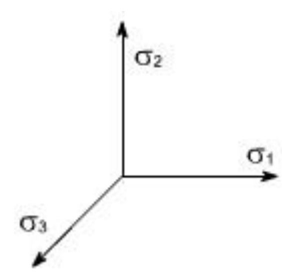
$$\in_3 = \frac{1}{E}\left[\sigma_3 - \gamma(\sigma_1 + \sigma_2)\right]$$

Thus, the failure criterion becomes

$$\left(\frac{\sigma_1}{E} - \gamma\frac{\sigma_2}{E} - \gamma\frac{\sigma_3}{E}\right) = \frac{\sigma_{yp}}{E}$$

or

$$\boxed{\sigma_1 - \gamma\sigma_2 - \gamma\sigma_3 = \sigma_{yp}}$$



(d) Total strain energy per unit volume theory :

The theory assumes that the failure occurs when the total strain energy for a complex state of stress system is equal to that at the yield point a tensile test.

$$\frac{1}{2E}\left[\sigma_1^{\,2} + \sigma_2^{\,2} + \sigma_3^{\,2} - 2\gamma(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right] = \frac{\sigma_{yp}^2}{2E}$$

Therefore, the failure criterion becomes $$\sigma_1^{\,2} + \sigma_2^{\,2} + \sigma_3^{\,2} - 2\gamma(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = \sigma_{yp}^2$$

It may be noted that this theory gives fair by good results for ductile materials.

(e) Maximum shear strain energy per unit volume theory :

This theory states that the failure occurs when the maximum shear strain energy component for the complex state of stress system is equal to that at the yield point in the tensile test.

$$\frac{1}{12G}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] = \frac{\sigma_{yp}^2}{6G}$$

Where G = shear modulus of regidity

Hence the criterion for the failure becomes $$\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] = 2\sigma_{yp}^2$$

As we know that a general state of stress can be broken into two components i.e,

(i) Hydrostatic state of stress (the strain energy associated with the hydrostatic state of stress is known as the volumetric strain energy)

(ii) Distortional or Deviatoric state of stress (The strain energy due to this is known as the shear strain energy)

As we know that the strain energy due to distortion is given as

$$U_{distortion} = \frac{1}{12G}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$

This is the distortion strain energy for a complex state of stress, this is to be equaled to the maximum distortion energy in the simple tension test. In order to get we may assume that one of the principal stress say (σ_1) reaches the yield point (σ_{yp}) of the material. Thus, putting in above equation $\sigma_2 = \sigma_3 = 0$ we get distortion energy for the simple test i.e

$$U_d = \frac{2\sigma_1^2}{12G}$$

Futher $\sigma_1 = \sigma_{yp}$

Thus, $\boxed{U_d = \frac{\sigma_{yp}^2}{6G}}$ for a simple tension test.

Source: http://nptel.ac.in/courses/Webcourse-contents/IIT-ROORKEE/strength%20of%20materials/homepage.htm