TENSORS: STRESS, STRAIN AND ELASTICITY

Introduction

Many physical properties of crystalline materials are direction dependent because the arrangement of the atoms in the crystal lattice are different in different directions. If one heats a block of glass it will expand by the same amount in each direction, but the expansion of a crystal will differ depending on whether one is measuring parallel to the a-axis or the b-axis. For this reason properties such as the elasticity and thermal expansivity cannot be expressed as scalars. We use tensors as a tool to deal with more this complex situation and because single crystal properties are important for understanding the bulk behavior of rocks (and Earth), we wind up dealing with tensors fairly often in mineral physics.

What is a Tensor

A tensor is a multi-dimensional array of numerical values that can be used to describe the physical state or properties of a material. A simple example of a geophysically relevant tensor is stress. Stress, like pressure is defined as force per

unit area. Pressure is isotropic, but if a material has finite strength, it can support different forces applied in different directions.

Figure 1 below, illustrates a unit cube of material with forces acting on it in three dimensions. By dividing by the surface area over which the forces are acting, the stresses on the cube can be obtained. Any arbitrary stress state can be decomposed into 9 components (labeled σ_{ij}). These components form a second rank tensor; the stress tensor (Figure 1).

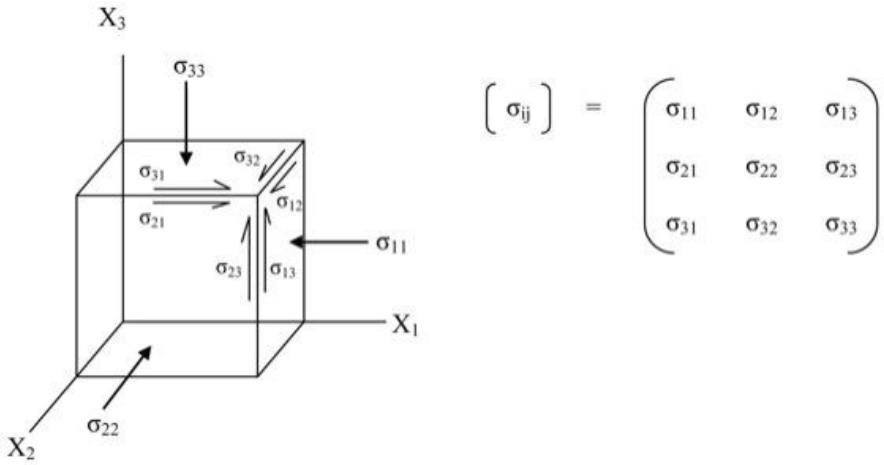

Figure 1

Tensor math allows us to solve problems that involve tensors. For example, let's say you measure the forces imposed on a single crystal in a deformation apparatus. It is easy to calculate the values in the stress tensor in the coordinate system tied to the apparatus. However you may be really interested in understanding the stresses

acting on various crystallographic planes, which are best viewed in terms of the crystallographic coordinates.

Tensor math allows you to calculate the stresses acting on the crystallographic planes by transforming the stress tensor from one coordinate system to another. Another familiar tensor property is electrical permittivity which gives rise to birefringence in polarized light microscopy. You are probably familiar with the optical indicatrix which is an ellipsoid constructed on the three principle refractive indices. The refractive index in any given direction through the crystal is governed by the dielectric constant K_{ij} which is a tensor. The dielectric constants "maps" the electric field E_j into the electric displacement D_i :

$$\begin{pmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{pmatrix} (k_o) \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = \begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix}$$

Were $\mathbf{k_0}$ is the permitivity of a vacuum. D_i can be calculated from E_j as follows:

$D_1 = k_oK_{11}E_1 + k_oK_{12}E_2 + k_oK_{13}E_3$

$D_2 = k_oK_{21}E_1 + k_oK_{22}E_2 + k_oK_{23}E_3$

$D_3 = k_oK_{31}E_1 + k_oK_{32}E_2 + k_oK_{33}E_3$

So you can see that even if E_1 is the only non-zero value in the electric field, all the components of D_i may be non-zero.

Rank of a Tensor

Tensors are referred to by their "rank" which is a description of the tensor's dimension. A zero rank tensor is a scalar, a first rank tensor is a vector; a one-dimensional array of numbers. A second rank tensor looks like a typical square matrix. Stress, strain, thermal conductivity, magnetic susceptibility and electrical permittivity are all second rank tensors. A third rank tensor would look like a three-dimensional matrix; a cube of numbers. Piezoelectricity is described by a third rank tensor. A fourth rank tensor is a four-dimensional array of numbers. The elasticity of single crystals is described by a fourth rank tensor.

Tensor transformation

As mentioned above, it is often desirable to know the value of a tensor property in a new coordinate system, so the tensor needs to be "transformed" from the original coordinate system to the new one. As an example we will consider the transformation of a first rank tensor; which is a vector. If we have a vector P with components p_1, p_2, p_3 along the coordinate axes X_1, X_2, X_3 and we want to write P in terms of p'_1, p'_2, p'_3 along new coordinate axes Z_1, Z_2, Z_3, we first need to describe how the coordinate systems are related to each other.

This can be done by noting the angle between each axis of the new coordinate system and each axis of the new coordinate system; altogether there will be 9 angles, three of which are illustrated in Figure 2:

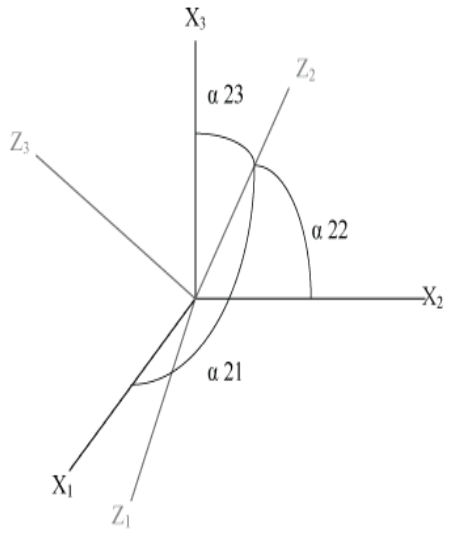

Figure 2

We can then express p'_1, p'_2,and p'_3 in terms of p_1, p_2, and p_3:

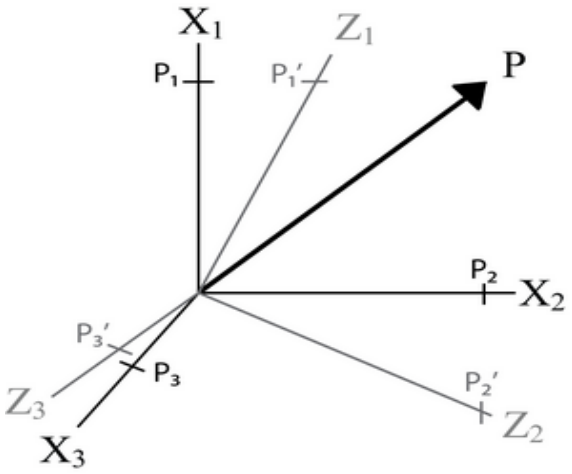

Figure 3

$$p'_1 = p_1 \cos \alpha_{11} + p_2 \cos \alpha_{12} + p_3 \cos \alpha_{13}$$

$$p'_2 = p_1 \cos \alpha_{21} + p_2 \cos \alpha_{22} + p_3 \cos \alpha_{23}$$

$$p'_3 = p_1 \cos \alpha_{31} + p_2 \cos \alpha_{32} + p_3 \cos \alpha_{33}$$

Source: https://serc.carleton.edu/NAGTWorkshops/mineralogy/mineral_physics/tensors.html