Stress transformation and Mohr's circle for stresses

1.1 General State of stress

Consider a certain body, subjected to external force. The force F is acting on the surface over an area dA of the surface. Then the stress is defined as the intensity with which the force is acting.

We can write the stress at a point as:

$\sigma = \lim_{dA \to 0} dF/dA$ = F/A if the stress is uniform

In general, the force acting is acting on a surface along some arbitrary direction can be resolved into components acting perpendicular to the given surface and also parallel to the given surface. Each component of force divided by the area over which it acts gives rise to three different states of stress on the surface, namely, one normal stress and two tangential stresses. These three stresses can be thought of acting along the three conventional x, y, z axes.

The state of stress on a 3dimensional object can be specified with 9 stress components. 3 normal stresses and 6 tangential (shear) stresses. This is diagrammed below:

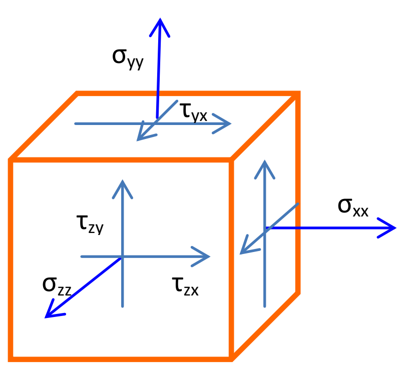

Fig. 1.1.1: Description of three dimensional state of stress

The 6 shear stress components, due to the requirement of zero rotation of the element or for moment balance, reduce to three shear stress components.

For satisfying moment balance, we have $\tau_{xy} = \tau_{yx}$and so on.

Normal stresses are represented with repeated subscripts, the first subscript represents the direction and the second subscript represents the plane on which it is acting. In case of shear stress, the first subscript represents the plane on which it is acting and the second subscript represents the direction along which the shear stress is acting.we can interchange the definition for the two subscripts of stress.

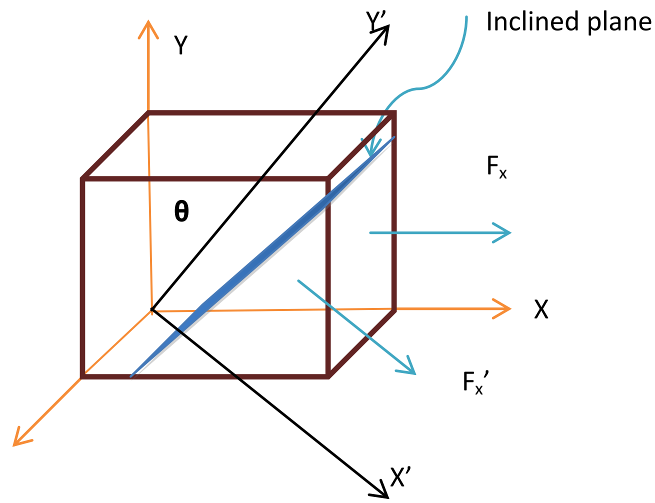

Fig.1.1.2: Stress on an inclined plane

Consider the force F_x acting on the right face of the cube, along x axis, as shown in diagram above. Consider a plane inclined with an area of A', as shown shaded. The normal to the shaded plane Y' is inclined at angle theta with Y axis.

F_x' the force acting along X' direction can be written as:

$F_x' = F_x \cos\boldsymbol{\theta}$

Now the stress along X' direction, normal to a plane inclined at angle **θ**with xy plane is

$\sigma_x' = \frac{Fx'}{Ax'} = \frac{F_x}{Ax/cos\theta} cos\theta = \sigma_x cos^2\theta$

Similarly, we can write the shear stress on the inclined plane as:

$$\tau_{xy}' = \tau_{xy} sin\theta cos\theta$$

In the same manner one can obtained the expression for the transformed stress σ_y' as:

$$\sigma_y' = \sigma_y cos\theta sin\theta$$

We note from the above expressions that the transformed stresses involve sine and cosine functions of the angle of rotation of the axes.

We can generalize the expressions for transformed stresses by writing:

$$\sigma_{ij} = \sum_{n=1}^{3} \sum_{m=1}^{n} l_{im}\, l_{jn} \sigma_{mn}$$

l_{im} is the direction cosine of angle between the axes I and m.

Triaxial state of stress may be rarely come across, in applications such as thick walled pressure vessels. In metal forming operations, triaxial state of stress is rarely come across. Therefore, matters get simplified with some assumptions. The first important assumption is plane stress condition.

1.2 Plane stress:

Many metal forming processes involve biaxial state of stress.

If one of the three normal and shear stresses acting on a body is zero, the condition of stress is called plane stress condition.All stresses act parellel to x and y axes.

i.e. $\sigma_{zz} = 0$, $\tau_{xz} = 0$

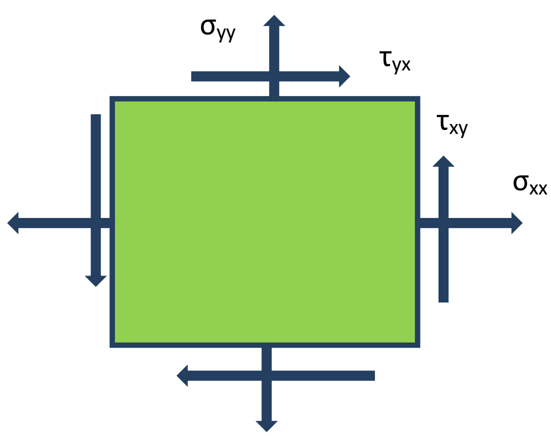

Fig. 1.2.1: Plane stress condition

Plane stress condition is come across in many engineering and forming applications.

When we consider crystalline solids, deformation is predominantly by slip of atomic planes of atoms along preferred directions.

Normally, slip can be easy if the shear stress acting on the slip planes is sufficiently high and acts along preferred slip direction. Slip planes may be inclined with respect to the external stress acting on solids. It becomes necessary to transform the stresses acting along the original axes into the inclined planes. Stress transformation becomes necessary in such cases.

1.3 Stress transformation in plane stress:

Consider the plane stress condition acting on a plane as shown. The stresses are to be transformed onto a plane which is inclined at an angle θ with respect to x, y axes.

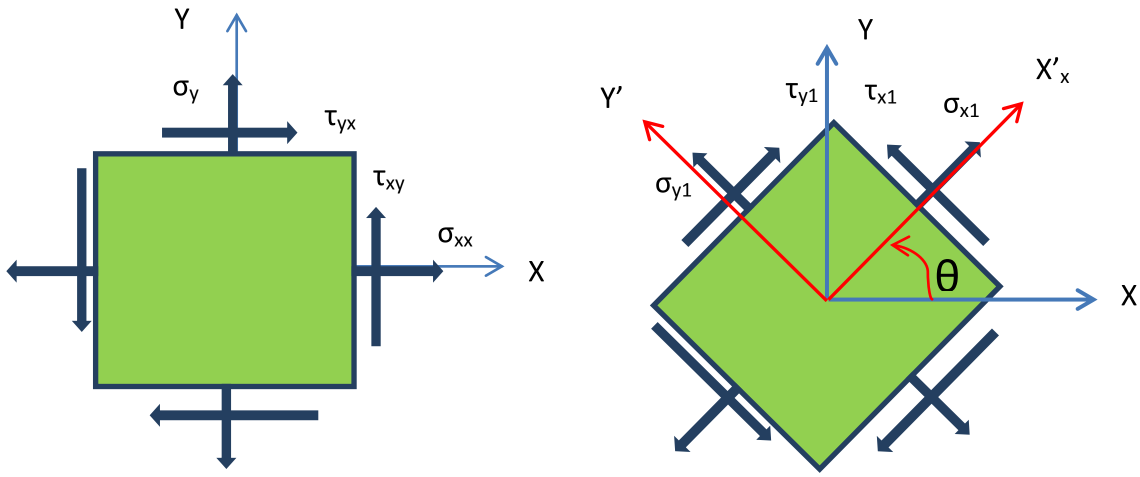

Fig. 1.3.1: Representation of stresses on inclined plane

Let X' and Y' be the new axes got by rotation of the x and y axes through the angle θ.

After the plane has been rotated about the z axis, the stresses acting on the plane along the new axes are to be obtained.

In order to obtain these transformed stresses, we take equilibrium of forces on the inclined plane both perpendicular to and parallel to the inclined plane. Or else, we can write the expression for transformed stress using the direction cosines:

$$\sigma_x' = l_{x'x}^2\sigma_x + \ l_{x'y}^2\sigma_y + 2l_{x'x}l_{x'y}\tau_{xy}$$

$$= 2cos^2\Theta\sigma_x + 2sin^2\Theta\sigma_y + 2cos\Theta sin\Theta\tau_{xy}$$

Similarly, we could write for the y' normal stress and shear stress.

The transformed stresses are given as:

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\sigma_{y1} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} cos2\theta - \tau_{xy}\, sin2\theta$$

And

$$\tau_{x_1 y_1} = \frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

Where σ_{x1} is the normal stress acting on the inclined plane and τ_{x1y1} is the shear stress acting on the inclined plane.

The above three equations are known as transformation equations for plane stress.

One is interested in maximum and minimum normal and shear stresses acting on the inclined plane in order to design components against failure.

The maximum normal stress and shear stress can be found by differentiating the stress transformation equations with respect to θ and equate to zero.

The maximum and minimum stresses are called principal stresses and the plane on which they act are called principal planes.

Maximum normal stress:

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + {\tau_{xy}}^2}$$

Maximum shear stress:

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + {\tau_{xy}}^2}$$

Also we find that $\tau_{max} = \frac{\sigma_1 - \sigma_2}{2}$

On a plane on which the principal normal stress acts, the shear stress is zero. Similarly, on a plane on which the principal shear is acting, the normal stresses are zero.

The angle corresponding to the principal planes can be obtained from:

$\tan 2\Theta = \frac{\tau_{xy}}{\frac{\sigma_x - \sigma_y}{2}}$ for the principal normal planes

And $\tan 2\Theta = -\frac{\tau_{xy}}{\frac{\sigma_x - \sigma_y}{2}}$ for principal shear plane

From this we find that the plane of maximum shear is oriented at an angle of 45^o with respect to the planes of maximum or minimum normal stresses.

1.4 Mohr's circle for plane stress:

Stress transformation equations can be represented in the form: $(x-h)^2 + Y^2 = R^2$ which represents the equation of a circle. h is the distance of center, R is radius of circle.

For plane stress condition, the equation for Mohr's circle is gives as:

$$(\sigma_{x1} - \frac{\sigma_x + \sigma_y}{2})^2 + \tau_{x1y1}{}^2 = (\frac{\sigma_x - \sigma_y}{2})^2 + \tau^2{}_{xy}$$

Here center of circle is located at a distance of $\sigma_{av} = (\sigma_x + \sigma_y)/2$from origin.

Transformed equations of stress are represented graphically by a circle called Mohr's circle. It can be used for determining graphically the transformed stresses on a new inclined plane.

Positive x-axis is chosen as normal stress axis. Negative y axis is chosen as positive shear axis.

Suppose the state of stress, both normal and shear (σ_x and τ_{xy}) on the two faces of a cube is known.

Centre of the circle is located at the average stress along the x axis. Then the known state of stress is represented by a point with σ_x and τ_{xy} as coordinates.Another point diametrically opposite to this point is located with the coordinates corresponding to the stresses acting on the face which is at 90 degree to the first face of the cube.

Now a circle is drawn with distance between both pints as diameter.

Stresses on an inclined plane can be represented on the circle if the angle of inclination is known. Twice the real angle of inclination is represented on the Mohr's circle.

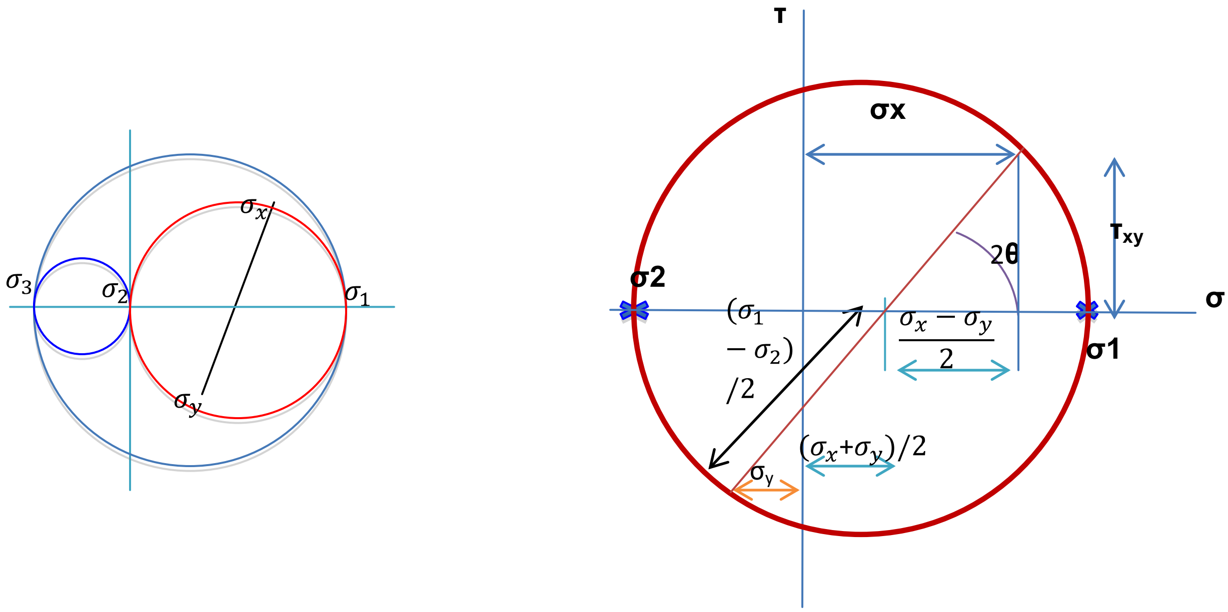

Fig. 1.4.1: Mohr's circle for triaxial and plane stresses

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