The Statics of Rigid Bodies

A material body can be considered to consist of a very large number of particles. A rigid body is one which does not deform, in other words the distance between the individual particles making up the rigid body remains unchanged under the action of external forces.

A new aspect of mechanics to be considered here is that a rigid body under the action of a force has a tendency to *rotate* about some axis. Thus, in order that a body be at rest, one not only needs to ensure that the resultant force is zero, but one must now also ensure that the forces acting on a body do not tend to make it rotate. This issue is addressed in what follows.

2.3.1 Moments, Couples and Equivalent Forces

When one swings a door on its hinges, it will move more easily if (i) one pushes hard, i.e. if the force is large, and (ii) if one pushes furthest from the hinges, near the edge of the door. It makes sense therefore to measure the rotational effect of a force on an object as follows:

The tendency of a force to make a rigid body rotate is measured by the **moment** of that force about an axis. The moment of a force **F** about an axis through a point o is defined as the product of the magnitude of **F** times the perpendicular distance *d* from the **line of action** of **F** and the axis o. This is illustrated in Fig. 2.3.1.

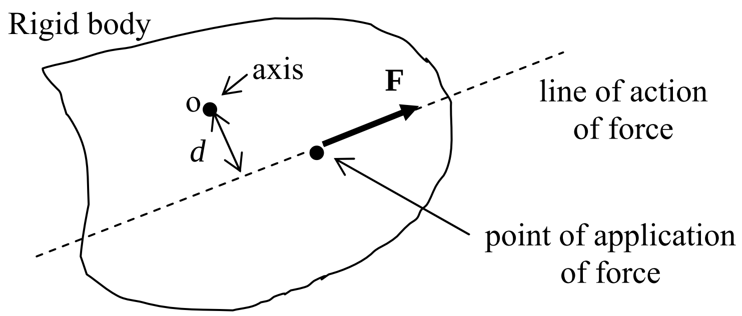

Figure 2.3.1: The moment of a force F about an axis o (the axis goes "into" the page)

The moment M_o of a force **F** can be written as

$$M_0 = Fd \tag{2.3.1}$$

Not only must the axis be specified (by the subscript *o*) when evaluating a moment, but the sense of that moment must be given; by convention, a tendency to rotate *counterclockwise* is taken to be a *positive* moment. Thus the moment in Fig. 2.3.1 is positive. The units of moment are the Newton metre (Nm)

Note that when the line of action of a force goes through the axis, the moment is zero.

It should be emphasized that there is not actually a physical axis, such as a rod, at the point *o* of Fig. 2.3.1; in this discussion, it is *imagined* that an axis is there.

Two forces of equal magnitude and acting along the same line of action have not only the same components F_x, F_y, but have equal moments about any axis. They are called **equivalent forces** since they have the same effect on a rigid body. This is illustrated in Fig. 2.3.2.

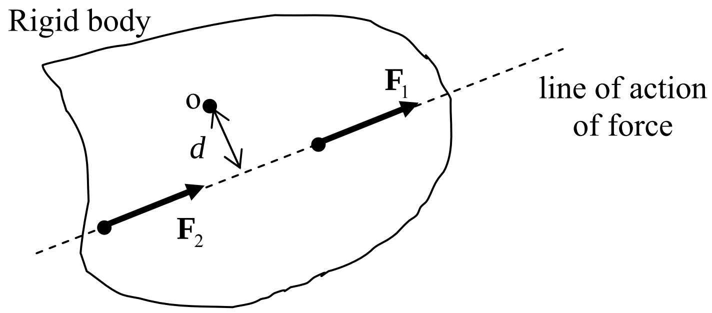

Figure 2.3.2: Two equivalent forces

Consider next the case of two forces of equal magnitude, parallel lines of action separated by distance *d*, and opposite sense. Any two such forces are said to form a **couple**. The only motion that a couple can impart is a rotation; unlike the forces of Fig. 2.3.2, the couple has no tendency to translate a rigid body. The moment of the couple of Fig. 2.3.3 about o is

$$M_o = Fd_2 - Fd_1 = Fd \qquad (2.3.2)$$

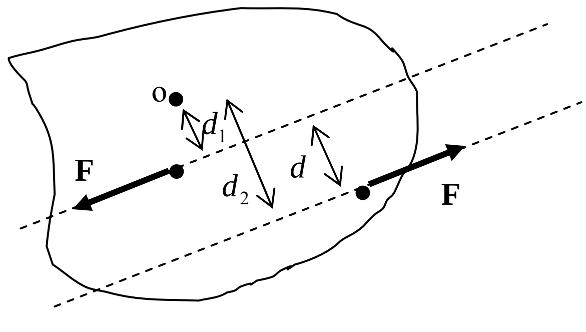

Figure 2.3.3: A couple

The sign convention which will be followed in most of what follows is that a couple is positive when it acts in a counterclockwise sense, as in Fig. 2.3.3.

It is straight forward to show the following three important properties of couples:
(a) the moment of Fig. 2.3.3 is also Fd about *any* axis in the rigid body, and so can be represented by *M*, without the subscript. In other words, this moment of the couple is independent of the choice of axis. {see ▲Problem 1}
(b) any two different couples having the same moment *M* are equivalent, in the sense that they tend to rotate the body in precisely the same way; it does not matter that the

forces forming these couples might have different magnitudes, act in different directions and have different distances between them.

(c) any two couples may be replaced by a single couple of moment equal to the algebraic sum of the moments of the individual couples.

Example

Consider the two couples shown in Fig. 2.3.4a. These couples can conveniently be represented schematically by semi-circular arrows, as shown in Fig. 2.3.4b. They can also be denoted by the letter *M*, the magnitude of their moment, since the magnitude of the forces and their separation is unimportant, only their product. In this example, if the body is in static equilibrium, the couples must be equal and opposite, $M_2 = -M_1$, i.e. the sum of the moments is zero and the net effect is to impart zero rotation on the body.

Note that the curved arrow for M_2 has been drawn counterclockwise, even though it is negative. It could have been illustrated as in Fig. 2.3.4c, but the version of 2.3.4b is preferable as it is more consistent and reduces the likelihood of making errors when solving problems (see later).

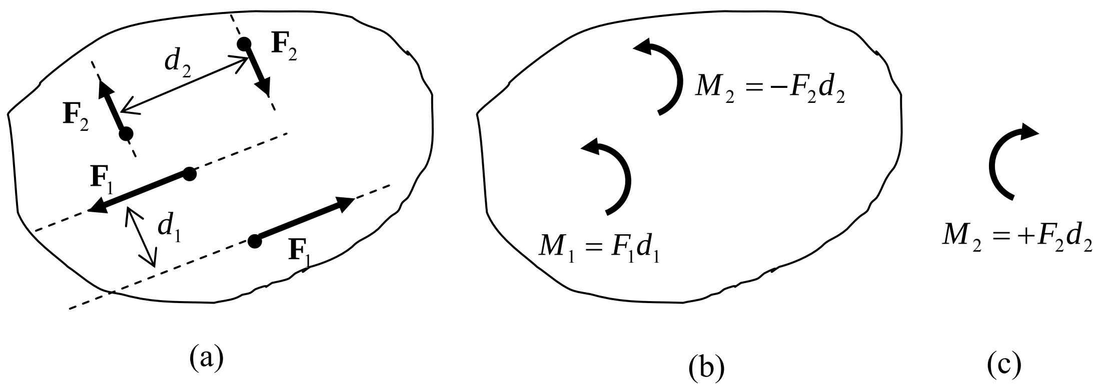

Figure 2.3.4: Two couples acting on a rigid body

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A final point to be made regarding couples is the following: any force is equivalent to (i) a force acting at any (other) point and (ii) a couple. This is illustrated in Fig. 2.3.5.

Referring to Fig. 2.3.5, a force **F** acts at position A. This force tends to translate the rigid body along its line of action and also to rotate it about any chosen axis. The system of forces in Fig. 2.3.5b are equivalent to those in Fig. 2.3.5a: a set of equal and opposite forces have simply been added at position B. Now the force at A and one of the forces at B form a couple, of moment *M* say. As in the previous example, the couple can conveniently be represented by a curved arrow, and the letter *M*. For illustrative purposes, the curved arrow is usually grouped with the force **F** at B, as shown in Fig. 2.3.5c. However, note that the curved arrow representing the moment of a couple, which can be placed anywhere and have the same effect, *is not associated with any particular point in the rigid body*.

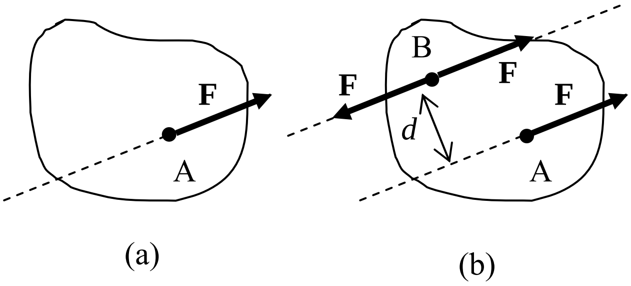


Figure 2.3.5: Equivalents force/moment systems; (a) a force F, (b) an equivalent system to (a), (c) an equivalent system involving a force and a couple *M*

Note that if the force at A was moved to a position other than B, the moment M of Fig. 2.3.5c would be different.

Example

Consider the spanner and bolt system shown in Fig. 2.3.6. A downward force of 200N is applied at the point shown. This force can be replaced by a force acting somewhere else, together with a moment. For the case of the force moved to the bolt-centre, the moment has the magnitude shown in Fig. 2.3.6b.

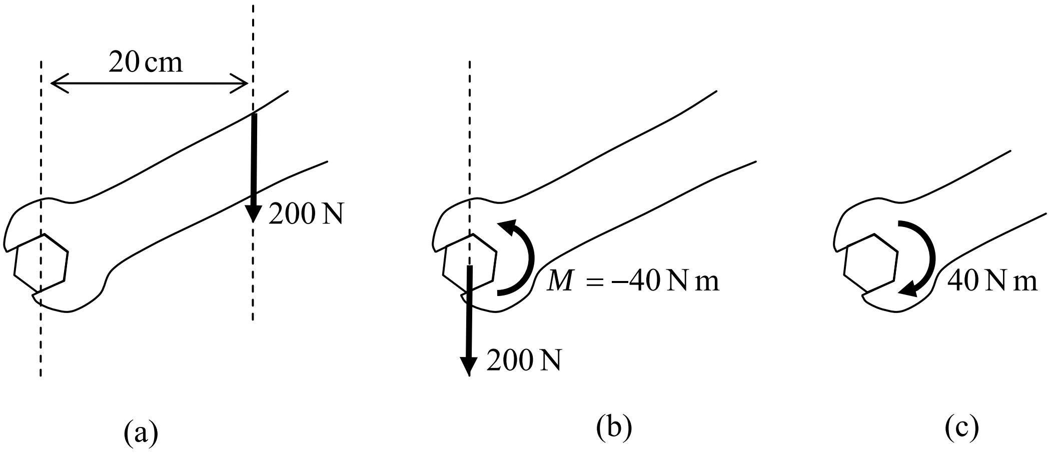


Figure 2.3.6: Equivalent force and force/moment acting on a spanner and bolt system

As mentioned, it is best to maintain consistency and draw the semi-circle representing the moment counterclockwise (positive) and given a value of -40 as in Fig. 2.3.6b; rather than as in Fig. 2.3.6c.

■

Example

Consider the plate subjected to the four external loads shown in Fig. 2.3.7a. An equivalent force-couple system **F**-M, with the force acting at the centre of the plate, can be calculated through

$$\sum F_x = 200\,\text{N}, \quad \sum F_y = 100\,\text{N}$$
$$\sum \text{M}_o = -(100)(100) - (50/\sqrt{2})(100) - (50/\sqrt{2})(100) + (200)(50) = -7071.07\,\text{Nmm}$$

and is shown in Fig. 2.3.7b. A **resultant force R** can also be derived, that is, an equivalent force positioned so that a couple is not necessary, as shown in Fig. 2.3.7.c.

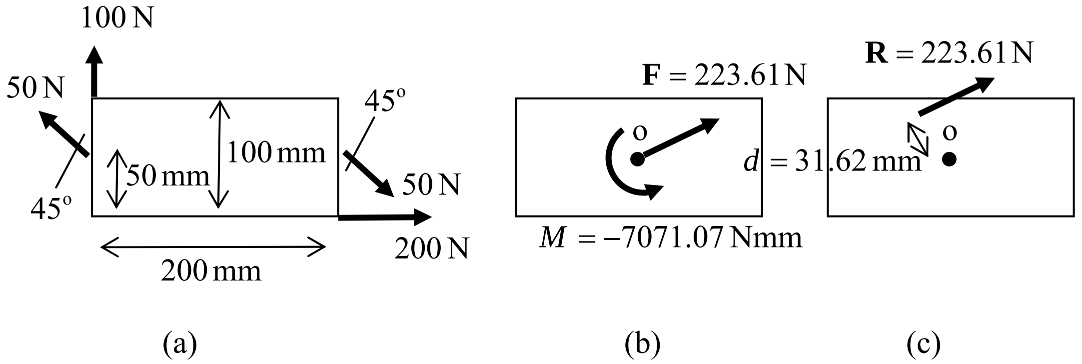


Figure 2.3.7: Forces acting on a plate; (a) individual forces, (b) an equivalent force-couple system at the plate-centre, (c) the resultant force

The force systems in the three figures are equivalent in the sense that they tend to impart (a) the same translation in the *x* direction, (b) the same translation in the *y* direction, and (c) the same rotation about *any* given point in the plate. For example, the moment about the upper left corner is

Fig 2.3.7a: $-(100)(0) - (50/\sqrt{2})(50) - (50/\sqrt{2})(150) + (200)(100)$
Fig 2.3.7b: $+(223.61)(89.44) - 7071$
Fig 2.3.7c: $+(223.61)(57.82)$

all leading to $M = 12928.93\,\text{Nmm}$ about that point.

■

2.3.2 Equilibrium of Rigid Bodies

The concept of equilibrium encountered earlier in the context of particles can now be generalized to the case of the rigid body:

Equilibrium of a Rigid Body
A rigid body is in equilibrium when the external forces acting on it form a system of forces equivalent to zero

The necessary and sufficient conditions that a (two dimensional) rigid body is in equilibrium are then

$$\sum F_x = 0, \quad \sum F_y = 0, \quad \sum M_o = 0 \quad \textbf{Equilibrium Equations (2D Rigid Body)} \quad (2.3.3)$$

that is, there is no resultant force and no resultant moment. Note that the $x-y$ axes and the axis of rotation o can be chosen completely arbitrarily: if the resultant force is zero, and the resultant moment about *one* axis is zero, then the resultant moment about *any* other axis in the body will be zero also.

2.3.3 Joints and Connections

Components in machinery, buildings etc., connect with each other and are supported in a number of different ways. In order to solve for the forces acting in such assemblies, one must be able to analyse the forces acting at such connections/supports.

One of the most commonly occurring supports can be idealised as a **roller support**, Fig. 2.3.8a. Here, the contacting surfaces are smooth and the roller offers only a normal reaction force (see §2.2.2). This reaction force is labelled $\mathbf{R}_y$, according to the conventional $x-y$ coordinate system shown. This is shown in the free-body diagram of the component.

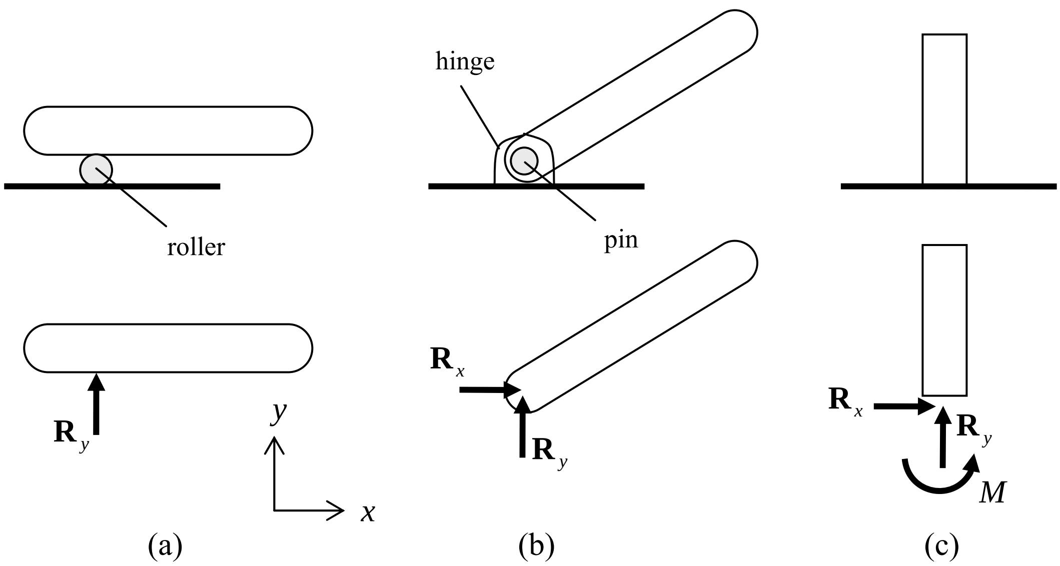


Figure 2.3.8: Supports and connections; (a) roller support, (b) pin joint, (c) clamped

Another commonly occurring connection is the **pin joint**, Fig. 2.3.8b. Here, the component is connected to a fixed hinge by a pin (going "into the page"). The component is thus constrained to move in one plane, and the joint does not provide resistance to this turning movement. The underlying support transmits a reaction force through the hinge pin to the component, which can have both normal ($\mathbf{R}_y$) and tangential ($\mathbf{R}_x$) components.

Finally, in Fig. 2.3.8c is shown a **fixed (clamped) joint**. Here the component is welded or glued and cannot move at the base. It is said to be **cantilevered**. The support in this case reacts with normal and tangential forces, but also with a couple of moment *M*, which resists any bending/turning at the base.

Example

For example, consider such a component loaded with a force **F** a distance *L* from the base, as shown in Fig. 2.3.9a. A free-body diagram of the component is shown in Fig. 2.3.9b. The known force **F** acts on the body and so do two unknown forces $\mathbf{R}_x$, $\mathbf{R}_y$, and a couple of moment *M*. The unknown forces and moment will be called **reactions** henceforth. If the component is static, the equilibrium equations 2.3.3 apply; one has, taking moments about the base of the component,

$$\sum F_x = F + R_x = 0, \quad \sum F_y = R_y = 0, \quad \sum M_{\mathrm{o}} = -FL + M = 0$$

and so

$$R_x = -F, \quad R_y = 0, \quad M = FL$$

The moment is positive and so acts in the direction shown in the Figure.

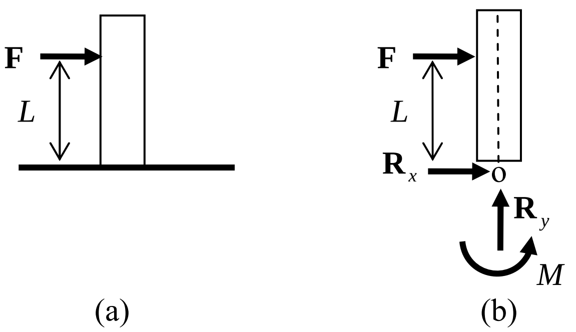


Figure 2.3.9: A loaded cantilevered component; (a) loaded component, (b) free body diagram of the component

The reaction moment of Fig. 2.3.9(b) can be experienced as follows: take a ruler and hold it firmly at one end, upright in your right hand. Simulate the applied force now by pushing against the ruler with a finger of your left hand. You will feel that, to maintain the ruler “vertical” at the base, you need to apply a twist with your right hand, in the direction of the moment shown in Fig. 2.3.9(b).

Note that, when solving this problem, moments were taken about the base. As mentioned already, one can take the moment about *any* point in the column. For example, taking the moment about the point where the force **F** is applied, one has

$$\sum M_{\mathrm{F}} = R_x L + M = 0$$

This of course leads to the same result as before, but the final calculation of the forces is now slightly more complicated; in general, it is easier if the axis is chosen to coincide with the point where the reaction forces act – this is because the reaction forces do not then appear in the moment equation: $\sum M_{\mathrm{o}} = -FL + M = 0$.

■

For ease of discussion, from now on, "couples" such as that encountered in Fig. 2.3.9 will simply be called "moments".

All the elements are now in place to tackle fairly complex static rigid body problems.

Example

Consider the plate subjected to the three external loads shown in Fig. 2.3.10a. The plate is supported by a roller at A and a pin-joint at B. The weight of the plate is assumed to be small relative to the applied loads and is neglected. A free body diagram of the plate is shown in Fig 2.3.10b. This shows all the forces acting *on* the plate. Reactions act at A and B: these forces represent the action of the base *on* the plate, preventing it from moving downward and horizontally. The equilibrium equations can be used to find the reactions:

$$\sum F_x = F_{xB} = 0 \quad \rightarrow \quad F_{xB} = 0$$
$$\sum F_y = +F_{yA} - 150 - 100 + 50 + F_{yB} = 0 \quad \rightarrow \quad F_{yA} + F_{yB} = 200\,\mathrm{N}$$
$$\sum M_{\mathrm{A}} = -(150)(50) - (100)(120) + (50)(200) + F_{yB}(200) = 0 \rightarrow \quad F_{yB} = 47.5\,\mathrm{N},$$

$$\rightarrow \mathrm{F}_{yA} = 152.5\,\mathrm{N}$$

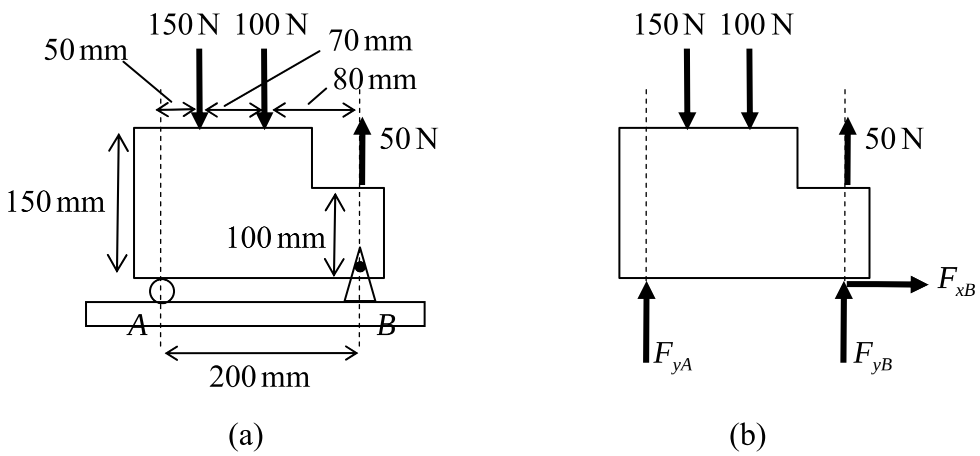


Figure 2.3.10: Equilibrium of a plate; (a) forces acting on the plate, (b) free-body diagram of the plate

The resultant moment was calculated by taking the moment about point A. As mentioned in relation to the previous example, one could have taken the moment about any other point in the plate. The "most convenient" point about which to take moments in this example would be point A or B, since in that case only one of the reaction forces will appear in the moment equilibrium equation.

■

In the above example there were three unknown reactions and three equilibrium equations with which to find them. If the roller was replaced with a pin, there would be four unknown reactions, and now there would not be enough equations with which to find the reactions. When this situation arises, the system is called **statically indeterminate**. To find the unknown reactions, one must relax the assumption of rigidity, and take into account the fact that all materials deform. By calculating deformations within the plate, the reactions can be evaluated. The deformation of materials is studied in the following chapters.

To end this Chapter, note the following:

(i) the equilibrium equations 2.3.3 result from Newton's laws, and are thus as valid for a body of water as they are for a body of hard steel; the external forces acting on a body of still water form a system of forces equivalent to zero.

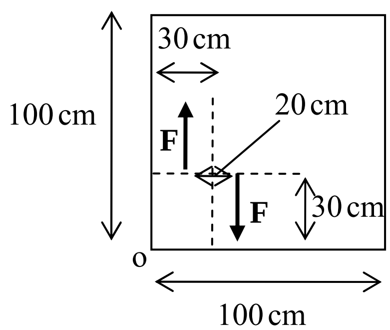
(ii) as mentioned already, Newton's laws apply not only to a complete body or structure, but to *any portion* of a body. The external forces acting *on* any free-body portion of static material form a system of forces equivalent to zero.

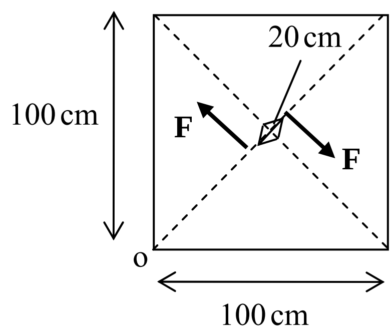
(iii) there is no such thing as a rigid body. Metals and other engineering materials can be considered to be "nearly rigid" as they do not deform by much under even fairly large loads. The analysis carried out in this Chapter is particularly relevant to these materials and in answering questions like: what forces act in the steel members of a suspension bridge under the load of self-weight and traffic? (which is just a more complicated version of the problem of Fig. 2.2.3 or Problem 3 below).

(iv) if the loads on the plate of Fig. 2.3.10a are too large, the plate will "break". The analysis carried out in this Chapter cannot answer where it will break or when it will break. The more sophisticated analysis carried out in the following Chapters is necessary to deal with this and many other questions of material response.

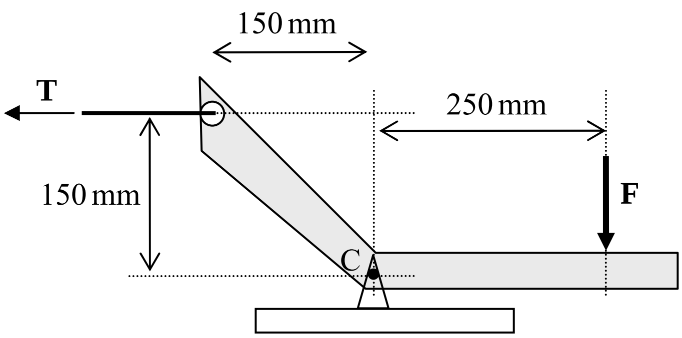
2.3.4 Problems

1. A plate is subjected to a couple Fd , with $d = 20\text{cm}$, as shown below left. Verify that the couple can be moved to the position shown below right, and the effect on the plate is the same, by showing that the moment about point o in both cases is $M = -20F$.

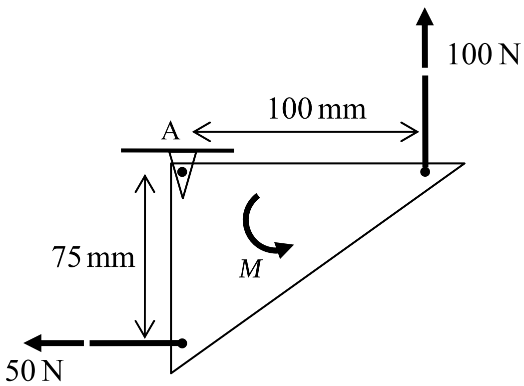




2. What force **F** must be applied to the following static component such that the tension in the cable, **T**, is 1kN? What are the reactions at the pin support C?



3. A machine part is hinged at A and subjected to two forces through cables as shown. What couple *M* needs to be applied to the machine part for equilibrium to be maintained? Where can this couple be applied?



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