

## Statics

Robots physically interact with the environment through mechanical contacts. Mating work pieces in a robotic assembly line, manipulating an object with a multi-fingered hand, and negotiating a rough terrain through leg locomotion are just a few examples of mechanical interactions. All of these tasks entail control of the contacts and interference between the robot and the environment. Force and moment acting between the robot end-effector and the environment must be accommodated for in order to control the interactions. In this chapter we will analyze the force and moment that act on the robot when it is at rest.

A robot generates a force and a moment at its end-effector by controlling individual actuators. To generate a desired force and moment, the torques of the multiple actuators must be coordinated. As seen in the previous chapter, the sensitivities of the individual actuators upon the end-effector motion, i.e. the Jacobian matrix, are essential in relating the actuator (joint) torques to the force and moment at the end-effector. We will obtain a fundamental theorem for force and moment acting on a multi degree-of-freedom robot, which we will find is analogous to the differential kinematics discussed previously.

### 6.1 Free Body Diagram

We begin by considering the free body diagram of an individual link involved in an open kinematic chain. Figure 6.1.1 shows the forces and moments acting on link  $i$ , which is connected to link  $i-1$  and link  $i+1$  by joints  $i$  and  $i+1$ , respectively. Let  $O_i$  be a point fixed to link  $i$  located on the joint axis  $i+1$  and  $O_{i-1}$  be a point fixed to link  $i-1$  on the joint axis  $i$ . Through the connections with the adjacent links, link  $i$  receives forces and moments from both sides of the link. Let  $\mathbf{f}_{i-1,i}$  be a three-dimensional vector representing the linear force acting from link  $i-1$  to link  $i$ . Likewise let  $\mathbf{f}_{i,i+1}$  be the force from link  $i$  to link  $i+1$ . The force applied to link  $i$  from link  $i+1$  is then given by  $-\mathbf{f}_{i,i+1}$ . The gravity force acting at the mass centroid  $C_i$  is denoted  $m_i\mathbf{g}$ , where  $m_i$  is the mass of link  $i$  and  $\mathbf{g}$  is the 3x1 vector representing the acceleration of gravity. The balance of linear forces is then given by

$$\mathbf{f}_{i-1,i} - \mathbf{f}_{i,i+1} + m_i\mathbf{g} = \mathbf{0}, \quad i = 1, \dots, n \quad (6.1.1)$$

Note that all the vectors are defined with respect to the base coordinate system  $O$ -xyz.

Next we derive the balance of moments. The moment applied to link  $i$  by link  $i-1$  is denoted  $\mathbf{N}_{i-1,i}$  and therefore the moment applied to link  $i$  by link  $i+1$  is  $-\mathbf{N}_{i,i+1}$ . Furthermore, the linear forces  $\mathbf{f}_{i-1,i}$  and  $-\mathbf{f}_{i,i+1}$  also cause moments about the centroid  $C_i$ . The balance of moments with respect to the centroid  $C_i$  is thus given by

$$\mathbf{N}_{i-1,i} - \mathbf{N}_{i,i+1} - (\mathbf{r}_{i-1,i} + \mathbf{r}_{i,C_i}) \times \mathbf{f}_{i-1,i} + (-\mathbf{r}_{i,C_i}) \times (-\mathbf{f}_{i,i+1}) = \mathbf{0}, \quad i = 1, \dots, n \quad (6.1.2)$$

where  $\mathbf{r}_{i-1,i}$  is the 3x1 position vector from point  $O_{i-1}$  to point  $O_i$  with reference to the base coordinate frame, and  $\mathbf{r}_{i,C_i}$  represents the position vector from point  $O_i$  to the centroid  $C_i$ .

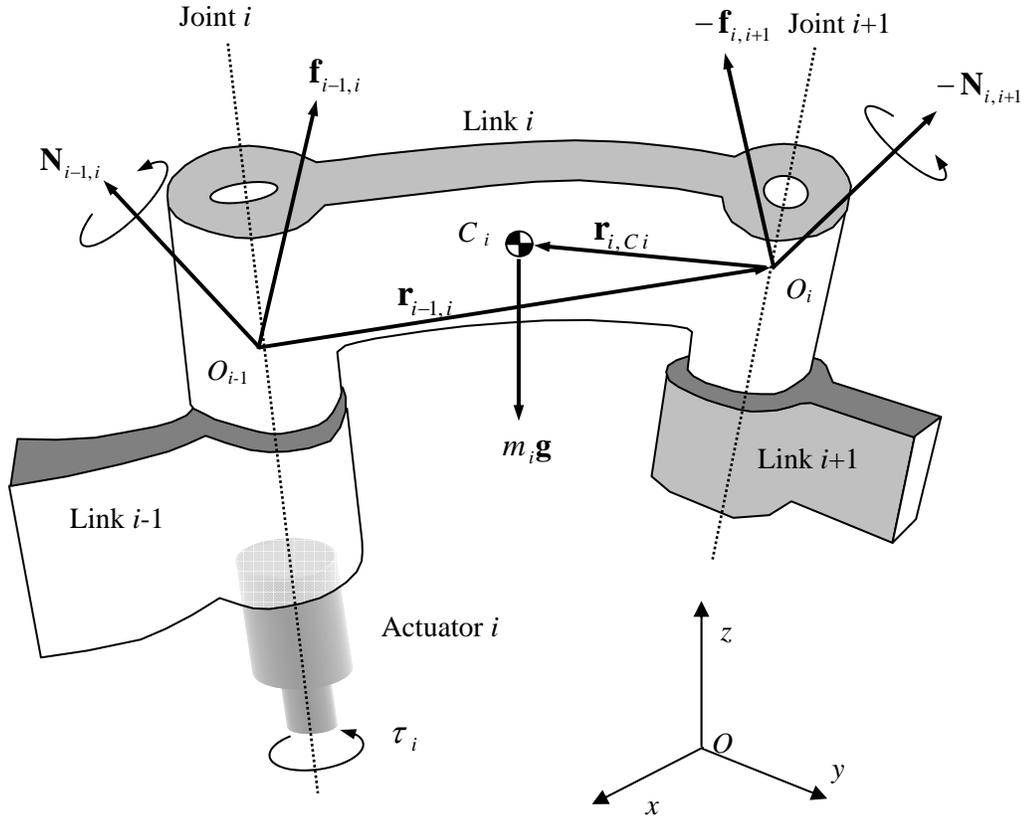


Figure 6.1.1 Free body diagram of the  $i$ -th link

The force  $\mathbf{f}_{i-1,i}$  and moment  $\mathbf{N}_{i-1,i}$  are called the coupling force and moment between the adjacent links  $i$  and  $i-1$ . For  $i=1$ , the coupling force and moment are  $\mathbf{f}_{0,1}$  and  $\mathbf{N}_{0,1}$ . These are interpreted as the reaction force and moment applied to the base link to which the arm mechanism is fixed. See Figure 6.1.2-(a). When  $i = n$ , on the other hand, the above coupling force and moment become  $\mathbf{f}_{n,n+1}$  and  $\mathbf{N}_{n,n+1}$ . As the end-effector, i.e. link  $n$ , contacts the environment, the reaction force acts on the end-effector. See Figure 6.1.2-(b). For convenience, we regard the environment as an additional link, numbered  $n+1$ , and represent the reaction force and moment by  $-\mathbf{f}_{n,n+1}$  and  $-\mathbf{N}_{n,n+1}$ , respectively.

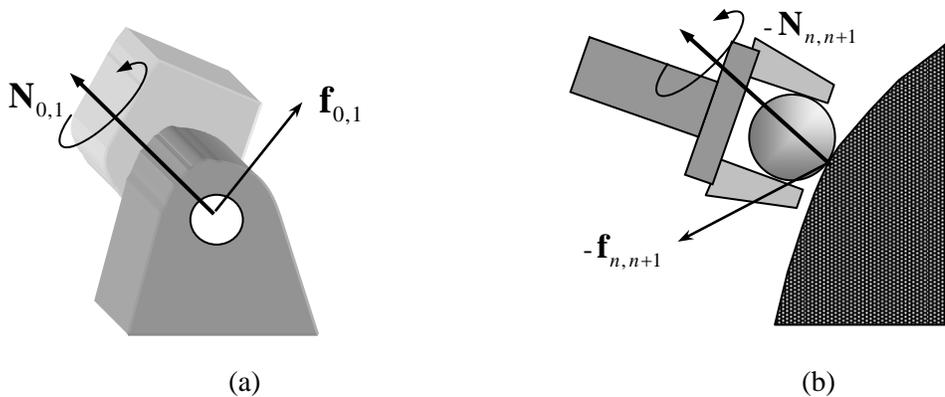


Figure 6.1.2 Force and moment that the base link exerts on link 1 (a), and the ones that the environment exerts on the end-effector, the final link (b)

The above equations can be derived for all the link members except for the base link, i.e.  $i=1,2, \dots, n$ . This allows us to form  $2n$  simultaneous equations of  $3 \times 1$  vectors. The number of coupling forces and moments involved is  $2(n+1)$ . Therefore two of the coupling forces and moments must be specified; otherwise the equations cannot be solved. The final coupling force and moment,  $\mathbf{f}_{n,n+1}$  and  $\mathbf{N}_{n,n+1}$ , are the force and moment that the end-effector applies to the environment. It is this pair of force and moment that the robot needs to accommodate in order to perform a given task. Thus, we specify this pair of coupling force and moment, and solve the simultaneous equations. For convenience we combine the force  $\mathbf{f}_{n,n+1}$  and the moment  $\mathbf{N}_{n,n+1}$ , to define the following six-dimensional vector:

$$\mathbf{F} = \begin{pmatrix} \mathbf{f}_{n,n+1} \\ \mathbf{N}_{n,n+1} \end{pmatrix} \quad (6.1.3)$$

We call the vector  $\mathbf{F}$  the endpoint force and moment vector, or the endpoint force for short.

## 6.2 Energy Method and Equivalent Joint Torques

In this section we will obtain the functional relationship between the joint torques and the endpoint force, which will be needed for accommodating interactions between the end-effector and the environment. Such a functional relationship may be obtained by solving the simultaneous equations derived from the free body diagram. However, we will use a different methodology, which will give an explicit formula relating the joint torques to the endpoint force without solving the simultaneous equations. The methodology we will use is the *energy method*, sometimes referred to as the *indirect method*. Since the simultaneous equations based on the balance of forces and moments are complex and difficult to solve, we will find that the energy method is the ideal choice when dealing with complex robotic systems.

In the energy method, we describe a system with respect to energy and work. Therefore, terms associated with forces and moments that do not produce, store, or dissipate energy are eliminated in its basic formula. In the free body diagram shown in Figure 6.1.1, many components of the forces and moments are so called “constraint forces and moments” merely joining adjacent links together. Therefore, constraint forces and moments do not participate in energy formulation. This significantly reduces the number of terms and, more importantly, will provide an explicit formula relating the joint torques to the endpoint force.

To apply the energy method, two preliminary formulations must be performed. One is to separate the net force or moment generating energy from the constraint forces and moments irrelevant to energy. Second, we need to find independent displacement variables that are geometrically admissible satisfying kinematic relations among the links.

Figure 6.2.1 shows the actuator torques and the coupling forces and moments acting at adjacent joints. The coupling force  $\mathbf{f}_{i-1,i}$  and moment  $\mathbf{N}_{i-1,i}$  are the resultant force and moment acting on the individual joint, comprising the constraint force and moment as well as the torque generated by the actuator. Let  $\mathbf{b}_{i-1}$  be the  $3 \times 1$  unit vector pointing in the direction of joint axis  $i$ , as shown in the figure. If the  $i$ -th joint is a revolute joint, the actuator generates joint torque  $\tau_i$  about the joint axis. Therefore, the joint torque generated by the actuator is one component of the coupling moment  $\mathbf{N}_{i-1,i}$  along the direction of the joint axis:

$$\tau_i = \mathbf{b}_{i-1}^T \cdot \mathbf{N}_{i-1,i} \quad (6.2.1)$$

For a prismatic joint, such as the  $(i+1)$ -st joint illustrated in Figure 6.2.1, the actuator generates a linear force in the direction of the joint axis. Therefore, it is the component of the linear coupling force  $\mathbf{f}_{i-1,i}$  projected onto the joint axis.

$$\tau_i = \mathbf{b}_{i-1}^T \cdot \mathbf{f}_{i-1,i} \quad (6.2.2)$$

Note that, although we use the same notation as that of a revolute joint, the scalar quantity  $\tau_i$  has the unit of a linear force for a prismatic joint. To unify the notation we use  $\tau_i$  for both types of joints, and call it a joint torque regardless the type of joint.

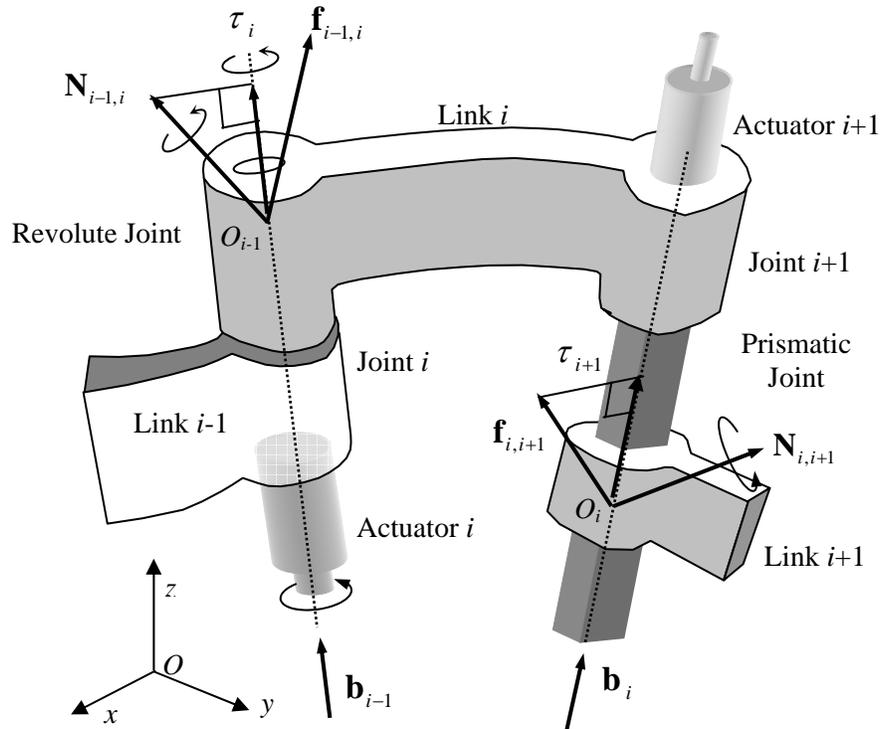


Figure 6.2.1 Joint torques as components of coupling force and moment

We combine all the joint torques from joint 1 through joint  $n$  to define the  $n \times 1$  joint torque vector:

$$\boldsymbol{\tau} = (\tau_1 \quad \tau_2 \quad \cdots \quad \tau_n)^T \quad (6.2.3)$$

The joint torque vector collectively represents all the actuators' torque inputs to the linkage system. Note that all the other components of the coupling force and moment are borne by the mechanical structure of the joint. Therefore, the constraint forces and moments irrelevant to energy formula have been separated from the net energy inputs to the linkage system.

In the free body diagram, the individual links are *disjointed*, leaving constraint forces and moments at both sides of the link. The freed links are allowed to move in any direction. In the energy formulation, we describe the link motion using independent variables alone. Remember that in a serial link open kinematic chain joint coordinates  $\mathbf{q} = (q_1 \quad \cdots \quad q_n)^T$  are a complete and independent set of generalized coordinates that uniquely locate the linkage system with independent variables. Therefore, these variables conform to the geometric and kinematic constraints. We use these joint coordinates in the energy-based formulation.

The explicit relationship between the  $n$  joint torques and the endpoint force  $\mathbf{F}$  is given by the following theorem:

**Theorem 6.1**

Consider an  $n$  degree-of-freedom, serial link robot having no friction at the joints. The joint torques  $\boldsymbol{\tau} \in \mathcal{R}^{n \times 1}$  that are required for bearing an arbitrary endpoint force  $\mathbf{F} \in \mathcal{R}^{6 \times 1}$  are given by

$$\boldsymbol{\tau} = \mathbf{J}^T \cdot \mathbf{F} \quad (6.2.4)$$

where  $\mathbf{J}$  is the  $6 \times n$  Jacobian matrix relating infinitesimal joint displacements  $d\mathbf{q}$  to infinitesimal end-effector displacements  $d\mathbf{p}$ :

$$d\mathbf{p} = \mathbf{J} \cdot d\mathbf{q} \quad (6.2.5)$$

□

Note that the joint torques in the above expression do not account for gravity and friction. They are the *net* torques that balances the endpoint force and moment. We call  $\boldsymbol{\tau}$  of eq.(3) the equivalent joint torques associated with the endpoint force  $\mathbf{F}$ .

**Proof**

We prove the theorem by using the *Principle of Virtual Work*. Consider virtual displacements at individual joints,  $\delta\mathbf{q} = (\delta q_1, \dots, \delta q_n)^T$ , and at the end-effector,  $\delta\mathbf{p} = (\delta\mathbf{x}_e^T, \delta\phi_e^T)^T$ , as shown in Figure 6.2.2. Virtual displacements are arbitrary infinitesimal displacements of a mechanical system that conform to the geometric constraints of the system. Virtual displacements are different from actual displacements, in that they must only satisfy *geometric* constraints and do not have to meet other laws of motion. To distinguish the virtual displacements from the actual displacements, we use the Greek letter  $\delta$  rather than the roman  $d$ .

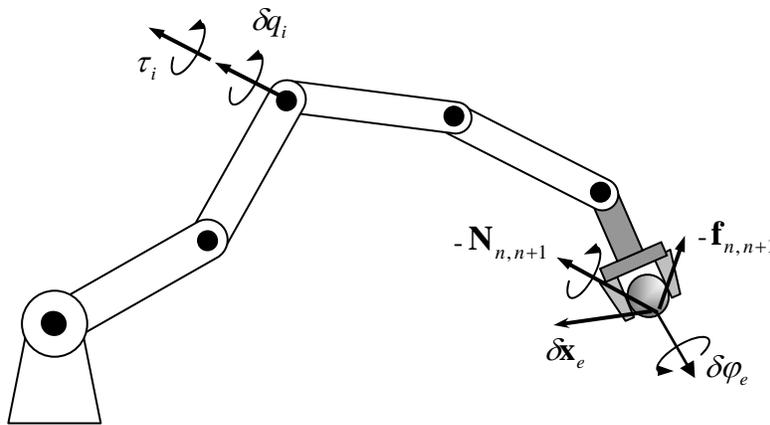


Figure 6.2.2 Virtual displacements of the end effector and individual joints

We assume that joint torques  $\boldsymbol{\tau} = (\tau_1 \ \tau_2 \ \dots \ \tau_n)^T$  and endpoint force and moment,  $-\mathbf{F}$ , act on the serial linkage system, while the joints and the end-effector are moved in the directions geometrically admissible. Then, the virtual work done by the forces and moments is given by

$$\begin{aligned}\delta Work &= \tau_1 \cdot \delta q_1 + \tau_2 \cdot \delta q_2 + \cdots + \tau_n \cdot \delta q_n - \mathbf{f}_{n,n+1}^T \cdot \delta \mathbf{x}_e - \mathbf{N}_{n,n+1}^T \cdot \delta \mathbf{p}_e \\ &= \boldsymbol{\tau}^T \delta \mathbf{q} - \mathbf{F}^T \delta \mathbf{p}\end{aligned}\quad (6.2.6)$$

According to the principle of virtual work, the linkage system is in equilibrium if, and only if, the virtual work  $\delta Work$  vanishes for arbitrary virtual displacements that conform to geometric constraints. Note that the virtual displacements  $\delta \mathbf{q}$  and  $\delta \mathbf{p}$  are not independent, but are related by the Jacobian matrix given in eq.(5). The kinematic structure of the robot mechanism dictates that the virtual displacements  $\delta \mathbf{p}$  is completely dependent upon the virtual displacement of the joints,  $\delta \mathbf{q}$ . Substituting eq.(5) into eq.(6) yields

$$\delta Work = \boldsymbol{\tau}^T \delta \mathbf{q} - \mathbf{F}^T \mathbf{J} \cdot \delta \mathbf{q} = (\boldsymbol{\tau} - \mathbf{J}^T \mathbf{F})^T \cdot \delta \mathbf{q}\quad (6.2.7)$$

Note that the vector of the virtual displacements  $\delta \mathbf{q}$  consists of all independent variables, since the joint coordinates of an open kinematic chain are generalized coordinates that are complete and independent. Therefore, for the above virtual work to vanish for arbitrary virtual displacements we must have:

$$\boldsymbol{\tau} = \mathbf{J}^T \mathbf{F}$$

This is eq.(6.2.4), and the theorem has been proven. □

The above theorem has broad applications in robot mechanics, design, and control. We will use it repeatedly in the following chapters.

### Example 6.1

Figure 6.2.3 shows a two-dof articulated robot having the same link dimensions as the previous examples. The robot is interacting with the environment surface in a horizontal plane. Obtain the equivalent joint torques  $\boldsymbol{\tau} = (\tau_1, \tau_2)^T$  needed for pushing the surface with an endpoint force of  $\mathbf{F} = (F_x, F_y)^T$ . Assume no friction.

The Jacobian matrix relating the end-effector coordinates  $x_e$  and  $y_e$  to the joint displacements  $\theta_1$  and  $\theta_2$  has been obtained in the previous chapter:

$$\mathbf{J} = \begin{pmatrix} -l_1 \sin \theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{pmatrix}\quad (5.1.8)$$

From **Theorem 6.1**, the equivalent joint torques are obtained by simply taking the transpose of the Jacobian matrix.

$$\begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} = \begin{pmatrix} -l_1 \sin \theta_1 - l_2 \sin(\theta_1 + \theta_2) & l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ -l_2 \sin(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{pmatrix} \cdot \begin{pmatrix} F_x \\ F_y \end{pmatrix}\quad (6.2.8)$$

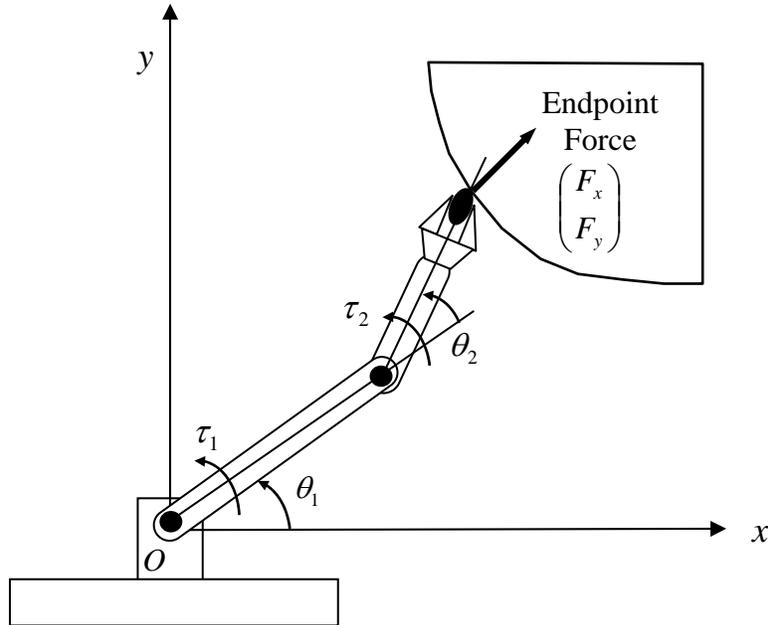


Figure 6.2.3 Two-dof articulated robot pushing the environment surface

### 6.3 Duality of Differential Kinematics and Statics

We have found that the equivalent joint torques are related to the endpoint force by the Jacobian matrix, which is the same matrix that relates the infinitesimal joint displacements to the end-effector displacement. Thus, the static force relationship is closely related to the differential kinematics. In this section we discuss the physical meaning of this observation.

To interpret the similarity between differential kinematics and statics, we can use the linear mapping diagram of Figure 5.4.1. Recall that the differential kinematic equation can be regarded as a linear mapping when the Jacobian matrix is fixed at a given robot configuration. Figure 6.3.1 reproduces Figure 5.4.1 and completes it with a similar diagram associated with the static analysis. As before, the range space  $R(\mathbf{J})$  represents the set of all the possible end-effector velocities generated by joint motions. When the Jacobian matrix is degenerate, or the robot configuration is singular, the range space does not span the whole vector space. Namely, there exists a direction in which the end-effector cannot move with a non-zero velocity. See the subspace  $S_2$  in the figure. The null space  $N(\mathbf{J})$ , on the other hand, represents the set of joint velocities that do not produce any velocity at the end-effector. If the null space contains a non-zero element, the differential kinematic equation has an infinite number of solutions that cause the same end-effector velocity.

The lower half of Figure 6.3.1 is the linear mapping associated with the static force relationship given by eq.(6.2.4). Unlike differential kinematics, the mapping of static forces is given by the transpose of the Jacobian, generating a mapping from the  $m$ -dimensional vector space  $\mathbf{V}^m$ , associated with the Cartesian coordinates of the end-effector, to the  $n$ -dimensional vector space  $\mathbf{V}^n$ , associated with the joint coordinates. Therefore the joint torques  $\boldsymbol{\tau}$  are always determined uniquely for any arbitrary endpoint force  $\mathbf{F}$ . However, for given joint torques, a balancing endpoint force does not always exist. As in the case of the differential kinematics, let us define the null space  $N(\mathbf{J}^T)$  and the range space  $R(\mathbf{J}^T)$  of the static force mapping. The null space  $N(\mathbf{J}^T)$  represents the set of all endpoint forces that do not require any torques at the joints to bear the corresponding load. In this case the endpoint force is borne entirely by the structure of the

linkage mechanism, i.e. constraint forces. The range space  $R(\mathbf{J}^T)$ , on the other hand, represents the set of all the possible joint torques that can balance the endpoint forces.

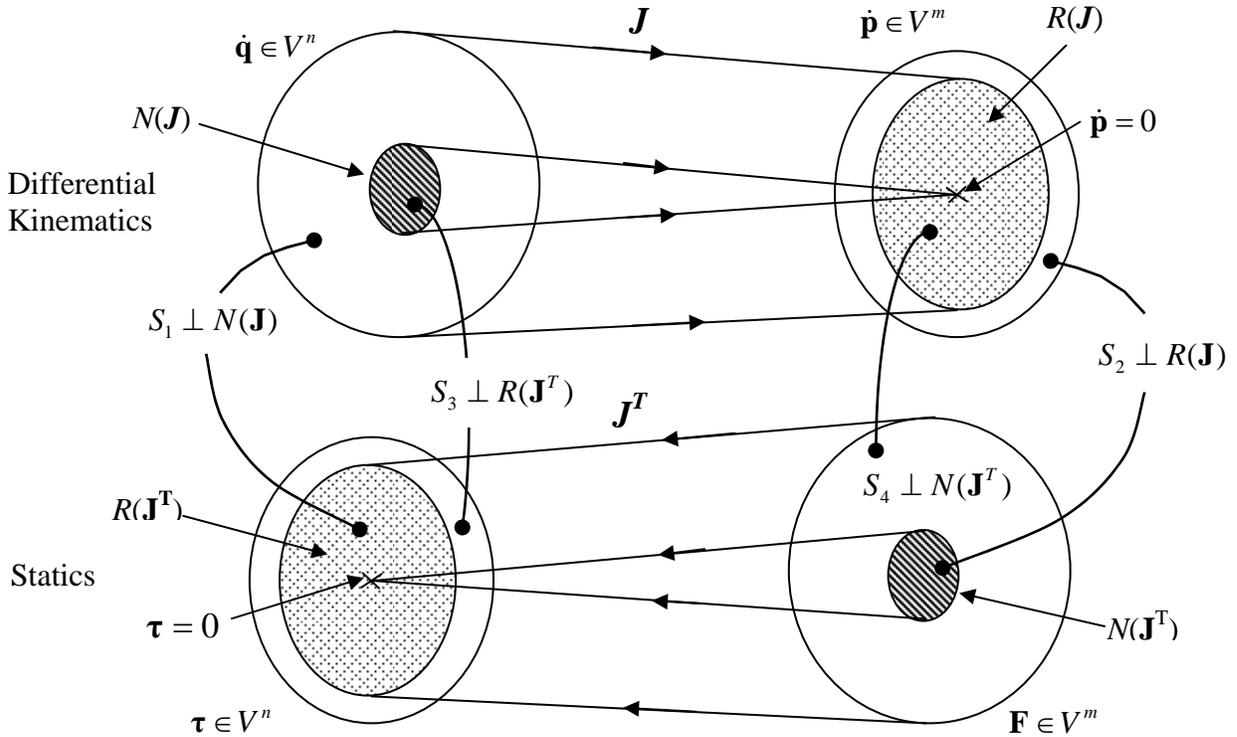


Figure 6.3.1 Duality of differential kinematics and statics

The ranges and null spaces of  $\mathbf{J}$  and  $\mathbf{J}^T$  are closely related. According to the rules of linear algebra, the null space  $N(\mathbf{J})$  is the orthogonal complement of the range space  $R(\mathbf{J}^T)$ . Namely, if a non-zero  $n$ -vector  $\mathbf{x}$  is in  $N(\mathbf{J})$ , it cannot also belong to  $R(\mathbf{J}^T)$ , and vice-versa. If we denote by  $S_1$  the orthogonal complement of  $N(\mathbf{J})$ , then the range space  $R(\mathbf{J}^T)$  is identical to  $S_1$ , as shown in the figure. Also, space  $S_3$ , i.e., the orthogonal complement of  $R(\mathbf{J}^T)$  is identical to  $N(\mathbf{J})$ . What this implies is that, in the direction in which joint velocities do not cause any end-effector velocity, the joint torques cannot be balanced with any endpoint force. In order to maintain a stationary configuration, the joint torques in this space must be zero.

There is a similar correspondence in the end-effector coordinate space  $V^m$ . The range space  $R(\mathbf{J})$  is the orthogonal complement to the null space  $N(\mathbf{J}^T)$ . Hence, the subspace  $S_2$  in the figure is identical to  $N(\mathbf{J}^T)$ , and the subspace  $S_4$  is identical to  $R(\mathbf{J})$ . Therefore, no joint torques are required to balance the end point force when the external force acts in the direction in which the end-effector cannot be moved by joint movements. Also, when the external endpoint force is applied in the direction along which the end-effector can move, the external force must be borne entirely by the joint torques. When the Jacobian matrix is degenerate or the arm is in a singular configuration, the null space  $N(\mathbf{J}^T)$  has a non-zero dimension, and the external force can be borne in part by the mechanical structure. Thus, differential kinematics and statics are closely related. This relationship is referred to as the *duality* of differential kinematics and statics.

## 6.4 Closed-Loop Kinematic Chains

The relationship between joint torques and the endpoint force obtained in Theorem 6.1 can be extended to a class of parallel-link mechanisms with closed kinematic-chains. It can also be extended to multi-fingered hands, leg locomotion, and other robot mechanisms having closed kinematic chains. In this section we discuss classes of closed kinematic chains based on the principle of virtual work.

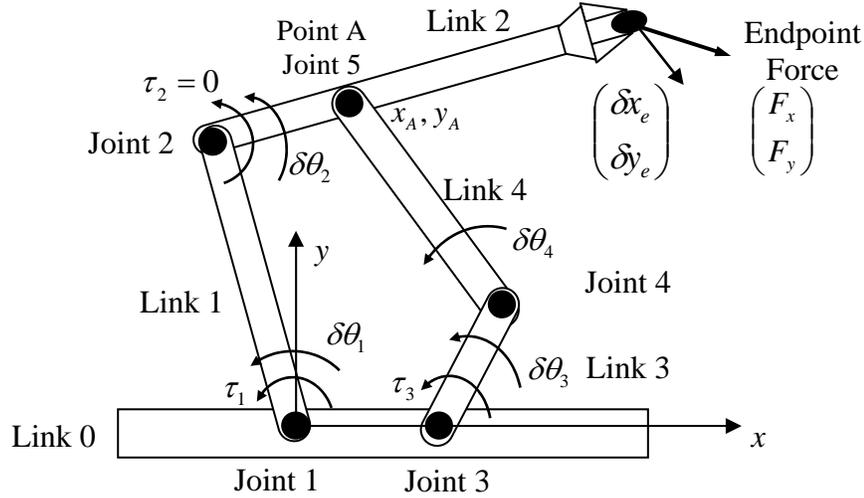


Figure 6.4.1 Five-bar-link robot exerting endpoint force

We begin by revisiting the five-bar-link planar robot shown in Figure 6.4.1. This robot has two degrees of freedom, comprising two active joints, Joints 1 and 3, and three passive joints, Joints 2, 4, and 5. Therefore the virtual work associated with the endpoint force and joint torques is given by

$$\delta Work = \tau_1 \delta \theta_1 + \tau_2 \delta \theta_2 + \tau_3 \delta \theta_3 + \cdots + \tau_5 \delta \theta_5 - F_x \delta x_e - F_y \delta y_e \quad (6.4.1)$$

We assume no friction at the joints. Therefore the three passive joints cannot bear any torque load about their joint axis. Substituting  $\tau_2 = \tau_4 = \tau_5 = 0$  into the above yields

$$\delta Work = (\tau_1 \quad \tau_3)^T \begin{pmatrix} \delta \theta_1 \\ \delta \theta_3 \end{pmatrix} - (F_x \quad F_y)^T \begin{pmatrix} \delta x_e \\ \delta y_e \end{pmatrix}. \quad (6.4.2)$$

For any given configuration of the robot, the virtual displacements of the end-effector are uniquely determined by the virtual displacements of Joints 1 and 3. In fact, the former is related to the latter via the Jacobian matrix:

$$\mathbf{J} = \begin{pmatrix} \frac{\partial x_e}{\partial \theta_1} & \frac{\partial x_e}{\partial \theta_3} \\ \frac{\partial y_e}{\partial \theta_1} & \frac{\partial y_e}{\partial \theta_3} \end{pmatrix} \quad (6.4.3)$$

Using this Jacobian,

$$\delta Work = \boldsymbol{\tau}^T \delta \mathbf{q} - \mathbf{F}^T \mathbf{J} \cdot \delta \mathbf{q} = (\boldsymbol{\tau} - \mathbf{J}^T \mathbf{F})^T \cdot \delta \mathbf{q} = 0, \quad \forall \delta \mathbf{q} \quad (6.4.4)$$

where

$$\delta \mathbf{q} = (\delta \theta_1 \quad \delta \theta_3)^T, \quad \delta \mathbf{p} = (\delta x_e \quad \delta y_e)^T \quad (6.4.5)$$

Eq.(4) implies

$$\boldsymbol{\tau} = \mathbf{J}^T \cdot \mathbf{F} \quad (6.4.6)$$

which is the same form as eq.(6.2.4).

In general the following Corollary holds.

**Corollary 6.1**

Consider an  $n$  degree-of-freedom robot mechanism with  $n$  active joints. Assume that all the joints are frictionless, and that, for a given configuration of the robot mechanism, there exists a unique Jacobian matrix relating the virtual displacements of its end-effector,  $\delta \mathbf{p} \in \mathcal{R}^{m \times 1}$ , to the virtual displacements of the active joints,  $\delta \mathbf{q} \in \mathcal{R}^{n \times 1}$ ,

$$\delta \mathbf{p} = \mathbf{J} \delta \mathbf{q}. \quad (6.4.7)$$

Then the equivalent joint torques  $\boldsymbol{\tau} \in \mathcal{R}^{n \times 1}$  to bear an arbitrary endpoint force  $\mathbf{F} \in \mathcal{R}^{m \times 1}$  is given by

$$\boldsymbol{\tau} = \mathbf{J}^T \cdot \mathbf{F} \quad (6.4.8)$$

□

Note that the joint coordinates associated with the active joints are not necessarily generalized coordinates that uniquely locate the system. For example, the arm configuration of the five-bar-link robot shown in Figure 6.4.1 is not uniquely determined with joint angles  $\theta_1$  and  $\theta_3$  alone. There are two configurations for given  $\theta_1$  and  $\theta_3$ . The corollary requires the differential relation to be defined uniquely in the vicinity of the given configuration.

## 6.5 Over-Actuated Systems

If an  $n$  degree-of-freedom robot system has more than  $n$  active joints, or less than  $n$  active joints, the above corollary does not apply. These are called *over-actuated* and *under-actuated systems*, respectively. Over-actuated systems are of particular importance in many manipulation and locomotion applications. In the following, we will consider the static relationship among joint torques and endpoint forces for a class of over-actuated systems.

Figure 6.5.1 shows a two-fingered hand manipulating an object within a plane. Note that both fingers are connected at the fingertips holding the object. While holding the object, the system has three degrees of freedom. Since each finger has two active joints, the total number of active joints is four. Therefore the system is over-actuated.

Using the notation shown in the figure, the virtual work is given by

$$\delta Work = \tau_1 \delta\theta_1 + \tau_2 \delta\theta_2 + \tau_3 \delta\theta_3 + \tau_4 \delta\theta_4 - F_x \delta x_e - F_y \delta y_e \quad (6.5.1)$$

Note that only three virtual displacements of the four joint angles are independent. There exists a differential relationship between one of the joints, say  $\theta_4$ , and the other three due to the kinematic constraint. Let us write it as

$$\delta\theta_4 = \mathbf{J}_c \cdot \delta\mathbf{q} \quad (6.5.2)$$

where  $\delta\mathbf{q} = (\delta\theta_1 \quad \delta\theta_2 \quad \delta\theta_3)^T$  are independent, and  $\mathbf{J}_c$  is the 1x3 Jacobian associated with the constraint due to the closed kinematic chain. Substituting this equation together with the Jacobian relating the end effector displacements to the tree joint displacements into eq.(1),

$$\delta Work = \boldsymbol{\tau}^T \delta\mathbf{q} + \tau_4 \mathbf{J}_c \delta\mathbf{q} - \mathbf{F}^T \mathbf{J} \delta\mathbf{q} = 0, \quad \forall \delta\mathbf{q} \quad (6.5.3)$$

The virtual work vanished for an arbitrary  $\delta\mathbf{q}$  only when

$$\boldsymbol{\tau} = -\mathbf{J}_c^T \tau_4 + \mathbf{J}^T \mathbf{F} \quad (6.5.4)$$

The two-fingered hand is at equilibrium only when the above condition is met. When the external endpoint force is zero:  $\mathbf{F}=\mathbf{0}$ , we obtain

$$\boldsymbol{\tau}_0 = -\mathbf{J}_c^T \tau_4 \quad (6.5.5)$$

This gives a particular combination of joint torques that do not influence the force balance with the external endpoint load  $\mathbf{F}$ . The joint torques having this particular proportion generate the internal force applied to the object, as illustrated in the figure. This internal force is a grasp force that is needed for performing a task.

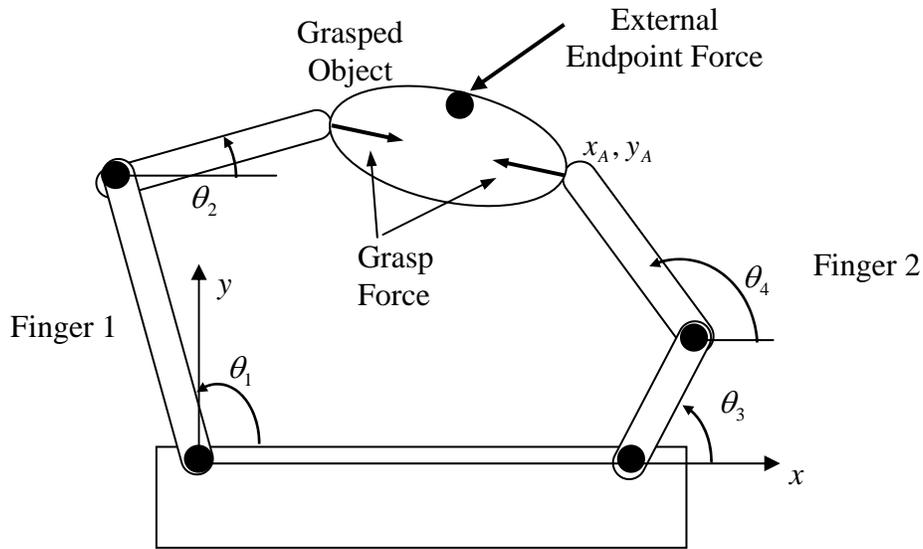


Figure 6.5.1 Two-fingered hand manipulating a grasped object

**Exercise 6.2**

Define geometric parameters needed in Figure 6.5.1, and obtain the two Jacobian matrices associated with the two-fingered hand holding an object. Furthermore, obtain the grasp force using the Jacobian matrices and the joint torques.

Source: <http://ocw.mit.edu/courses/mechanical-engineering/2-12-introduction-to-robotics-fall-2005/lecture-notes/chapter6.pdf>