

Relative Motion using Translating Axes

In the previous lectures we have described particle motion as it would be seen by an observer standing still at a fixed origin. This type of motion is called *absolute* motion. In many situations of practical interest, we find ourselves forced to describe the motion of bodies while we are simultaneously moving with respect to a fixed reference frame. There are many examples where such situations occur. The absolute motion of a passenger inside an aircraft is best described if we first consider the motion of the passenger relative to the aircraft, and then the motion of the aircraft relative to the ground. If we try to track the motion of aircraft in the airspace using satellites, it makes sense to first consider the motion of the aircraft relative to the satellite and then combine this motion with the motion of the satellite relative to the earth's surface. In this lecture we will introduce the ideas of *relative* motion analysis.

Types of observers

For the purpose of studying relative motion, we will consider four different types of observers (or reference frames) depending on their motion with respect to a fixed frame:

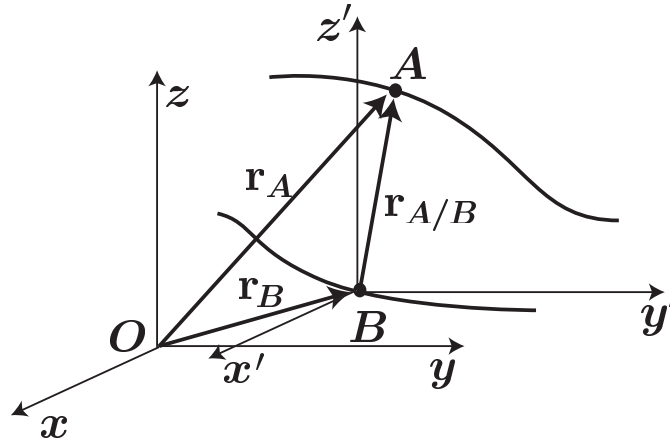
- observers who do not accelerate or rotate, i.e. those who at most have constant velocity.
- observers who accelerate but do not rotate
- observers who rotate but do not translate
- observers who accelerate and rotate

In this lecture we will consider the relative motion involving observers of the first two types, and defer the study of relative motion involving rotating frames to the next lecture.

Relative motion using translating axes

We consider two particles, A and B in curvilinear motions along two different paths. We describe their motion with respect to a fixed reference frame xyz with origin O and with unit vectors \mathbf{i} , \mathbf{j} and \mathbf{k} , as before, and call the motion relative to this frame absolute. The position of particle A is given by $\mathbf{r}_A(t)$ and the position of particle B is given by $\mathbf{r}_B(t)$; both vectors are defined with respect to the fixed reference frame O . In addition, it is useful in many problems to ask "how would B describe the motion of A and how would this description be translate to the fixed inertial coordinate system O ?"

In order to answer this question, we consider another *translating* reference frame attached to particle B , $x'y'z'$, with unit vectors \mathbf{i}' , \mathbf{j}' and \mathbf{k}' . Translating means that the angles between the axes xyz and $x'y'z'$ do not change during the motion.



In the figure, we have chosen, for convenience, the axes xyz to be parallel to the axes $x'y'z'$, but it should be clear that one could have non-parallel **translating** axes. (By our ground rules, these axis must not rotate, i.e the angles between them must not change; that possibility will be considered in a subsequent lecture.) The position vector $\mathbf{r}_{A/B}$ defines the position of A with respect to point B in the reference frame $x'y'z'$. The subscript notation “ A/B ” means “ A relative to B ”. The positions of A and B relative to the absolute frame are given by the vectors \mathbf{r}_A and \mathbf{r}_B , respectively. Thus, we have

$$\mathbf{r}_A = \mathbf{r}_B + \mathbf{r}_{A/B} .$$

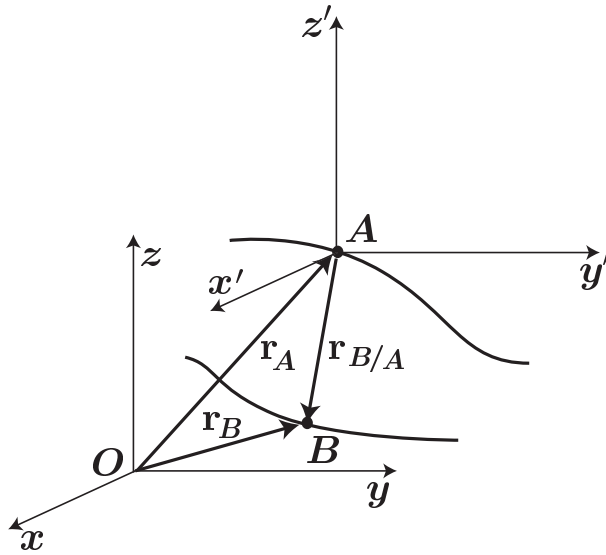
If we derive this expression with respect to time, we obtain

$$\dot{\mathbf{r}}_A = \dot{\mathbf{r}}_B + \dot{\mathbf{r}}_{A/B} \quad \text{or} \quad \mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B} ,$$

which relates the absolute velocities \mathbf{v}_A and \mathbf{v}_B to the relative velocity of A as observed by B . Differentiating again, we obtain an analogous expression for the accelerations,

$$\ddot{\mathbf{r}}_A = \ddot{\mathbf{r}}_B + \ddot{\mathbf{r}}_{A/B} \quad \text{or} \quad \mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B} .$$

We will now reverse the roles of A and B , asking “how would A describe the motion of B ”? We now attach the reference frame $x'y'z'$ to A , and then we can observe B from A . The relative motion of B as seen by A is now denoted $\mathbf{r}_{B/A}$, the position of B as seen by A .



The same arguments as before will give us,

$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A}, \quad \mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}, \quad \mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}.$$

Comparing these expressions with those above, we see that

$$\mathbf{r}_{B/A} = -\mathbf{r}_{A/B}, \quad \mathbf{v}_{B/A} = -\mathbf{v}_{A/B}, \quad \mathbf{a}_{B/A} = -\mathbf{a}_{A/B},$$

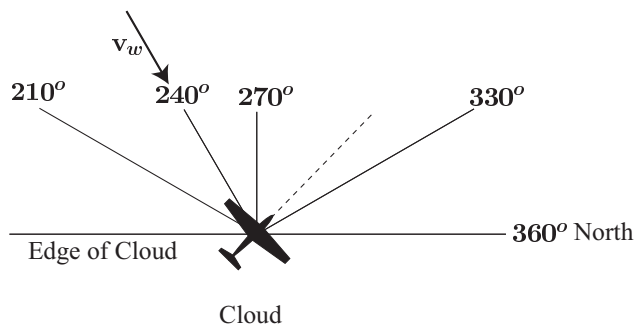
as expected.

One important observation is that, whenever the moving system, say A , has a constant velocity relative to the fixed system, O , then the acceleration seen by the two observers is the same, i.e., if $\mathbf{a}_A = \mathbf{0}$, then $\mathbf{a}_B = \mathbf{a}_{B/A}$. We shall see that this broadens the application of Newton's second law to systems which have a constant absolute velocity.

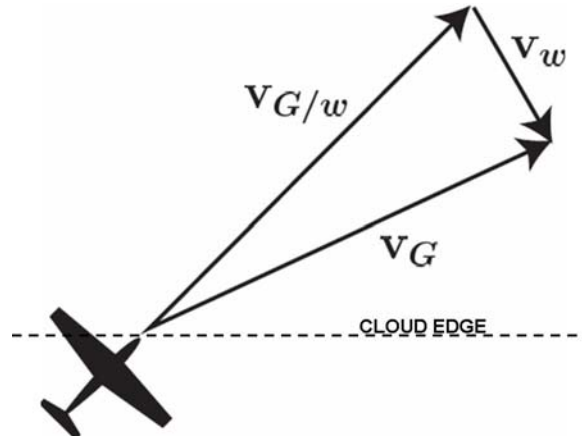
Example

Glider in cross wind

Consider a glider flying above the edge of a cloud which is aligned North/South (360°). The glider is flying horizontally, and the cloud is stationary with respect to the ground. At the altitude of the glider flight, there is a wind velocity, \mathbf{v}_w , of magnitude 52 knots coming from the direction 240° , as shown in the sketch.



The glider, on the other hand, is flying at a speed $v_{G/w}$ relative to the moving air mass and at an orientation given by his compass heading.

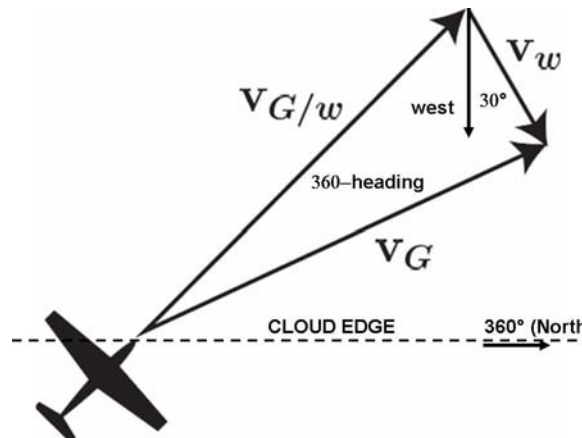


In order to determine the speed of the glider with respect to the ground, we consider a reference frame moving with the wind speed and write

$$v_G = v_w + v_{G/w},$$

For these conditions, to balance the effect of wind speed and glider motion so that the glider stays at the edge of the cloud, we require

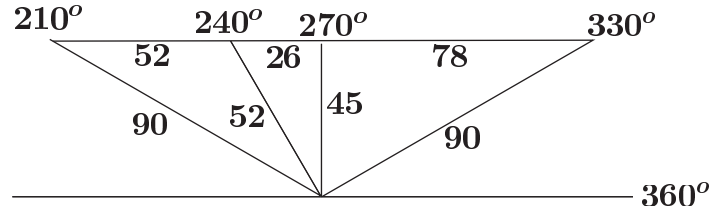
$$v_{G/w} * \sin(360^\circ - \text{heading}) = v_w * \cos(30^\circ) \tag{1}$$



Using this diagram, by adding and subtracting the vector components of the various velocities, we can ask questions such as: if the glider is flying relative to the wind ($v_{G/w}$) at a speed of 90 knots, what angle must it make with the ground—its absolute course heading—to stay at the edge of the cloud? the glider flies with a heading of 330° and a speed (relative to the wind) of 90 knots, it will stay at the edge of the cloud with a ground speed of $104 = (26 + 78)$ knots north;

By inspection of the diagram below, we can also verify the following situations: if the glider flies with a heading of 330° and a speed (relative to the wind) of 90 knots, it will stay at the edge of the cloud with a

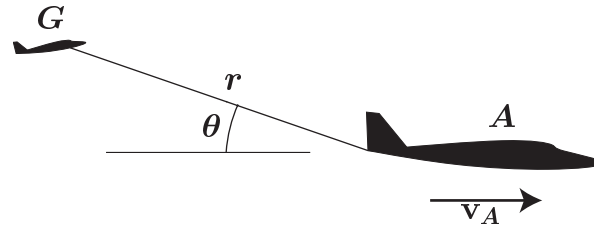
ground speed of $104 = (26 + 78)$ knots north; if, on the other hand, the glider flies with a heading of 210° at 90 knots, the glider will stay at the edge of the cloud with a ground speed of 52 knots south; if the glider flies into the wind at 240° , with a speed equal to 52 knots, it will remain stationary with respect to the ground; finally, the lowest speed at which the glider can fly to stay at the edge of the cloud is 45 knots and this will be possible for a heading of 270° . The ground speed in this case will be 26 knots.



Example

Aircraft towing glider (Meriam)

Airplane A is flying horizontally with a constant speed of 200km/h, and is towing a glider B . The glider is gaining altitude. The tow cable has a length $r = 60\text{m}$, and θ is increasing at a constant rate of 5 degrees per second. We want to determine the magnitude of the velocity, \mathbf{v}_G , and the acceleration, \mathbf{a}_G , of the glider for the instant when $\theta = 15^\circ$.



The velocity of the glider will be $\mathbf{v}_G = \mathbf{v}_A + \mathbf{v}_{G/A}$. The velocity of the glider relative to A is best expressed using a polar coordinate system. Thus, we write,

$$\mathbf{v}_{G/A} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta .$$

Since the length of the cable is constant, $v_r = \dot{r} = 0$, and $\dot{\theta} = 5(2\pi/360) = 0.0874$ rad/sec, which gives $v_\theta = r\dot{\theta} = 5.236$ m/s. We can now transform the velocity vector to cartesian coordinates to obtain,

$$\mathbf{v}_{G/A} = 5.236\mathbf{e}_\theta = 5.236 \sin 15^\circ\mathbf{i} + 5.236 \cos 15^\circ\mathbf{j} = 1.3552\mathbf{i} + 5.0575\mathbf{j} \text{ m/s} ,$$

or,

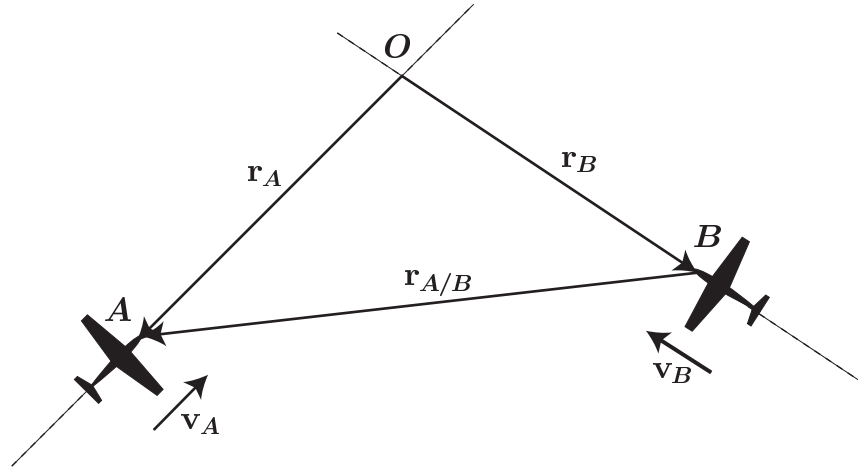
$$\mathbf{v}_G = ((200/3.6) + 1.3552)\mathbf{i} + 5.0575\mathbf{j} \text{ m/s}, \quad v_G = 57.135 \text{ m/s} = 205.7 \text{ km/h} .$$

For the acceleration $\mathbf{a}_G = \mathbf{a}_{G/A}$, since $\mathbf{a}_A = \mathbf{0}$. We also know that $\ddot{\theta} = 0$, and, obviously, $\ddot{r} = 0$, thus,

$$\mathbf{a}_{G/A} = -r\dot{\theta}^2\mathbf{e}_r = -0.457\mathbf{e}_r \text{ m/s}^2, \quad \mathbf{a}_G = -0.457 \text{ m/s}^2 .$$

Example**Aircraft collision avoidance**

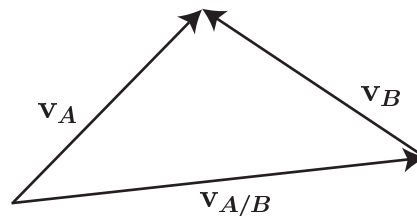
Consider two aircraft, A and B , flying horizontally at the same level with constant velocities \mathbf{v}_A and \mathbf{v}_B . Aircraft B is capable of determining by measurement the relative position and velocity of A relative to B , i.e., $\mathbf{r}_{A/B}$ and $\mathbf{v}_{A/B}$. We want to know whether it is possible to determine from this measurement whether the two aircraft are on a collision course.



Take the origin O at the hypothetical point of collision where the two trajectories intersect. If collision is to occur, the time taken for both aircraft to reach O will be the same. Therefore,

$$\frac{r_A}{v_A} = \frac{r_B}{v_B}.$$

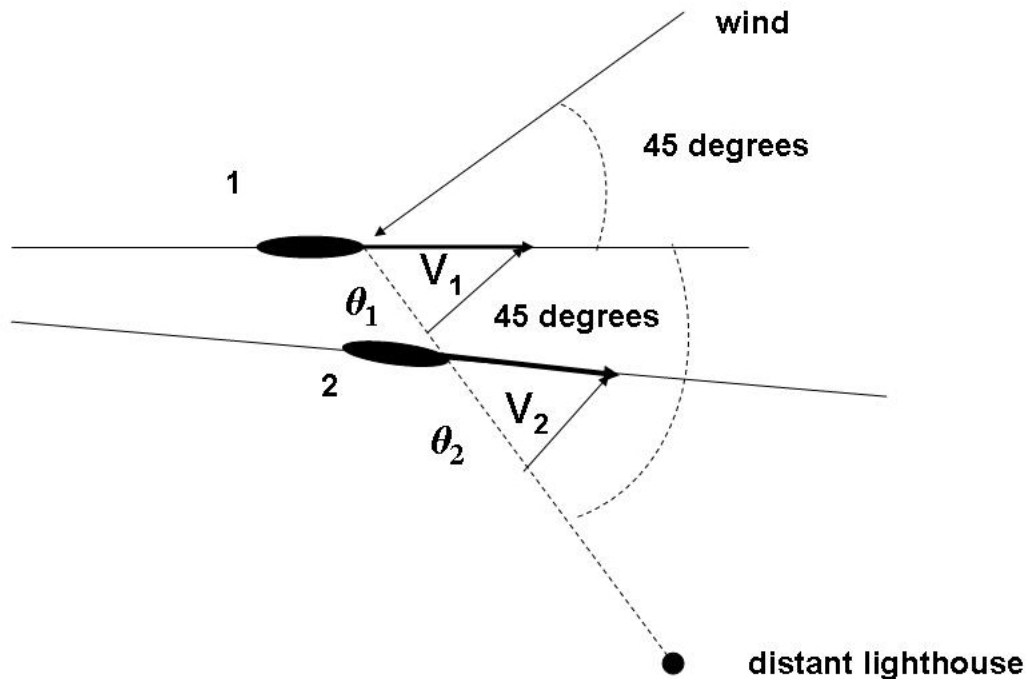
Now consider the velocity triangle formed by $\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$.



Clearly, \mathbf{v}_A is parallel to \mathbf{r}_A , and \mathbf{v}_B is parallel to \mathbf{r}_B . In addition, if collision is to occur, $\mathbf{v}_{A/B}$ will be parallel to $\mathbf{r}_{A/B}$, since the position and velocity triangles will, in this case, be similar. We see, therefore, that the condition for the two planes to collide is that the relative velocity between them is parallel to the relative position vector. If aircraft B uses a polar coordinate system (common for radar measurements), then the relative position of A with respect to B will be of the form $\mathbf{r}_{A/B} = r\mathbf{e}_r$. Therefore, the velocity also has to be of that form, which implies that $v_\theta = r\dot{\theta} = 0$, or $\theta = \text{constant}$. That is, the angle at which B sees A (sometimes called the bearing angle) should not change.

Example**Velocity made good to windward**

In sailboat racing, the most important variable is the component of velocity made by the sailboat in the direction of the wind. Typically sailboats can sail at an angle of 45° relative to the wind direction. Consider the sailboat 1 making a velocity vector v_1 at an angle of 45° with respect to the wind in the fixed coordinate system—he knows his angle relative to the wind from compass measurements. He is racing sailboat 2 who makes a velocity vector v_2 at an angle to the wind of θ_2 , neither of which is known to the skipper of sailboat 1.



The skipper of sailboat 1 would like to know whether he is traveling faster in the direction of the wind than sailboat 2. He sees a distant lighthouse at an angle of 45° from his direction of travel. He lines up his opponent with this stationary lighthouse. How can he use this observation to determine whether he is winning the race to the weather mark?

The instantaneous observation of relative velocity in the direction of the wind can be determined by whether sailboat 2 pulls ahead of the line between the lighthouse and a fixed point on sailboat 1. Sailboat 1 can observe the sign of $V_2 \cos \theta_2 - V_1 \cos \theta_1$ but not these quantities individually. However, that combination, which he can observe by sighting, is the difference in the velocities of the two sailboats made good in the direction of the wind. Therefore, if sailboat 2 drops behind the line of sight to the distant lighthouse, sailboat 1 is winning the race. However, if sailboat 2 pulls ahead of the line of sight to the distant lighthouse, sailboat 2 is winning the race.

ADDITIONAL READING

Source:<http://ocw.mit.edu/courses/aeronautics-and-astronautics/16-07-dynamics-fall-2009/lecture-notes/>

MIT16_07F09_Lec07.pdf