3.5 Plane Stress

This section is concerned with a special two-dimensional state of stress called **plane stress**. It is important for two reasons: (1) it arises in real components (particularly in thin components loaded in certain ways), and (2) it is a two dimensional state of stress, and thus serves as an excellent introduction to more complicated three dimensional stress states.

3.5.1 Plane Stress

The state of plane stress is defined as follows:

Plane Stress:
If the stress state at a material particle is such that the only non-zero stress components act in one plane only, the particle is said to be in plane stress.

The axes are usually chosen such that the $x-y$ plane is the plane in which the stresses act, Fig. 3.5.1.

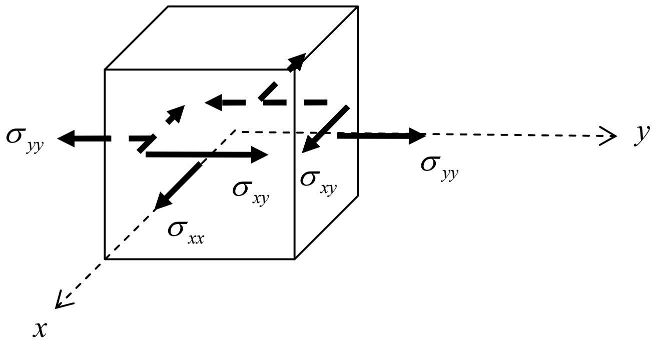


Figure 3.5.1: non-zero stress components acting in the $x-y$ plane

The stress can be expressed in the matrix form 3.4.1.

Example

The thick block of uniform material shown in Fig. 3.5.2, loaded by a constant stress σ_o in the x direction, will have $\sigma_{xx}=\sigma_0$ and all other components zero everywhere. It is therefore in a state of plane stress.

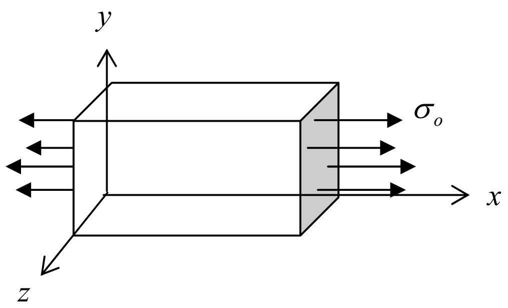


Figure 3.5.2: a thick block of material in plane stress

3.5.2 Analysis of Plane Stress

Next are discussed the **stress invariants**, **principal stresses** and **maximum shear stresses** for the two-dimensional plane state of stress, and tools for evaluating them. These quantities are useful because they tell us the complete state of stress at a point in simple terms. Further, these quantities are directly related to the strength and response of materials. For example, the way in which a material plastically (permanently) deforms is often related to the maximum shear stress, the directions in which flaws/cracks grow in materials is often related to the principal stresses, and the energy stored in materials is often a function of the stress invariants.

Stress Invariants

A stress invariant is some function of the stress components which is independent of the coordinate system being used; in other words, they have the same value no matter where the $x-y$ axes are drawn through a point. In a two dimensional space there are two stress invariants, labelled I_1 and I_2. These are

$$\boxed{\begin{aligned} I_1 &= \sigma_{xx} + \sigma_{yy} \\ I_2 &= \sigma_{xx}\sigma_{yy} - \sigma_{xy}^2 \end{aligned}} \qquad \textbf{Stress Invariants} \qquad (3.5.1)$$

These quantities can be proved to be invariant directly from the stress transformation equations, Eqns. 3.4.8 {▲Problem 1}. Physically, invariance of I_1 and I_2 means that they are the same for any chosen perpendicular planes through a material particle.

Combinations of the stress invariants are also invariant, for example the important quantity

$$\tfrac{1}{2}I_1 \pm \sqrt{\tfrac{1}{4}I_1^2 - I_2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\tfrac{1}{4}\left(\sigma_{xx} - \sigma_{yy}\right)^2 + \sigma_{xy}^2} \qquad (3.5.2)$$

Principal Stresses

Consider a material particle for which the stress, with respect to some $x-y$ coordinate system, is

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \tag{3.5.3}$$

The stress acting on different planes through the point can be evaluated using the Stress Transformation Equations, Eqns. 3.4.8, and the results are plotted in Fig. 3.5.3. The original planes are re-visited after rotating 180°.

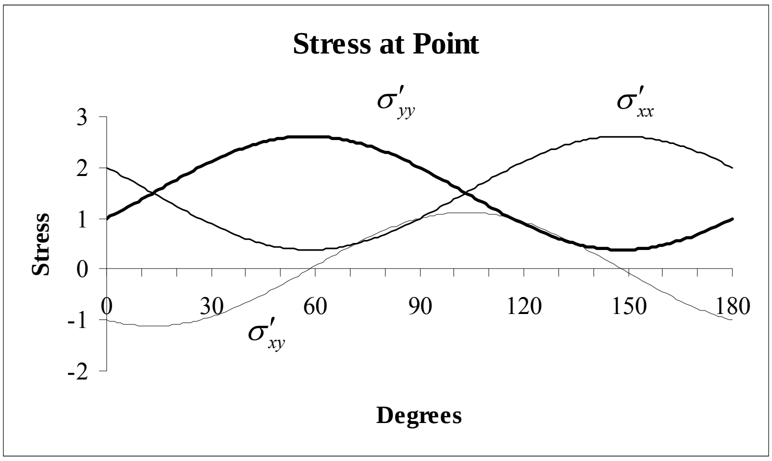


Figure 3.5.3: stresses on different planes through a point

It can be seen that there are two perpendicular planes for which the shear stress is zero, for θ ≈ 58° and $\theta \approx (58+90)^{\circ}$. In fact it can be proved that for every point in a material there are two (and only two) perpendicular planes on which the shear stress is zero (see below). These planes are called the **principal planes**. It will also be noted from the figure that the normal stresses acting on the planes of zero shear stress are either a maximum or minimum. Again, this can be proved (see below). These normal stresses are called principal stresses. The principal stresses are labelled σ_1 and σ_2, Fig. 3.5.4.

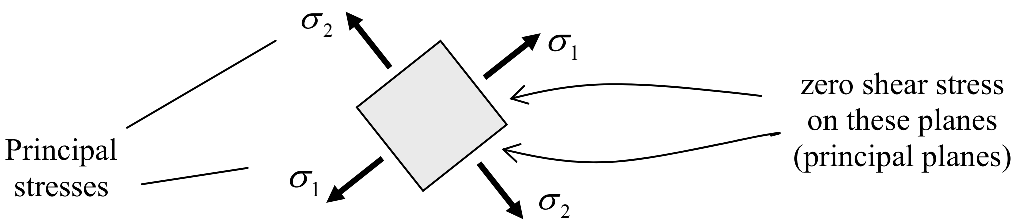


Figure 3.5.4: principal stresses

The principal stresses can be obtained by setting $\sigma'_{xy} = 0$ in the Stress Transformation Equations, Eqns. 3.4.8, which leads to the value of θ for which the planes have zero shear stress:

$$\boxed{\tan 2\theta = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}}} \qquad \textbf{Location of Principal Planes} \qquad (3.5.4)$$

For the example stress state, Eqn. 3.5.3, this leads to

$$\theta = \frac{1}{2}\arctan(-2)$$

and so the perpendicular planes are at $\theta = -31.72° \; (148.28°)$ and $\theta = 58.3°$.

Explicit expressions for the principal stresses can be obtained by substituting the value of θ from Eqn. 3.5.4 into the Stress Transformation Equations, leading to (see the Appendix to this section, §3.5.8)

$$\boxed{\begin{aligned} \sigma_1 &= \tfrac{1}{2}(\sigma_{xx} + \sigma_{yy}) + \sqrt{\tfrac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \sigma_{xy}^2} \\ \sigma_2 &= \tfrac{1}{2}(\sigma_{xx} + \sigma_{yy}) - \sqrt{\tfrac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \sigma_{xy}^2} \end{aligned}} \qquad \textbf{Principal Stresses} \qquad (3.5.5)$$

For the example stress state Eqn.3.5.3, one has

$$\sigma_1 = \frac{3 + \sqrt{5}}{2} \approx 2.62, \qquad \sigma_2 = \frac{3 - \sqrt{5}}{2} \approx 0.38$$

Note here that one uses the symbol σ_1 to represent the maximum principal stress and σ_2 to represent the minimum principal stress. By maximum, it is meant the algebraically largest stress so that, for example, $+1 > -3$.

From Eqns. 3.5.2, 3.5.5, the principal stresses are invariant; they are intrinsic features of the stress state at a point and do not depend on the coordinate system used to describe the stress state.

The question now arises: why are the principal stresses so important? One part of the answer is that the maximum principal stress is the largest normal stress acting on any plane through a material particle. This can be proved by differentiating the stress transformation formulae with respect to θ,

$$\begin{aligned}\frac{d\sigma'_{xx}}{d\theta} &= -\sin 2\theta(\sigma_{xx} - \sigma_{yy}) + 2\cos 2\theta\sigma_{xy} \\ \frac{d\sigma'_{yy}}{d\theta} &= +\sin 2\theta(\sigma_{xx} - \sigma_{yy}) - 2\cos 2\theta\sigma_{xy} \\ \frac{d\sigma'_{xy}}{d\theta} &= -\cos 2\theta(\sigma_{xx} - \sigma_{yy}) - 2\sin 2\theta\sigma_{xy}\end{aligned} \tag{3.5.6}$$

The maximum/minimum values can now be obtained by setting these expressions to zero. One finds that the normal stresses are a maximum/minimum at the very value of θ in Eqn. 3.5.4 – the value of θ for which the shear stresses are zero – the principal planes.

Very often the only thing one knows about the stress state at a point are the principal stresses. In that case one can derive a very useful formula as follows: align the coordinate axes in the principal directions, so

$$\sigma_{xx} = \sigma_1, \quad \sigma_{yy} = \sigma_2, \quad \sigma_{xy} = 0 \tag{3.5.7}$$

Using the transformation formulae with the relations $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$ and $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$ then leads to

$$\boxed{\begin{aligned}\sigma'_{xx} &= \frac{1}{2}(\sigma_1 + \sigma_2) + \frac{1}{2}(\sigma_1 - \sigma_2)\cos 2\theta \\ \sigma'_{yy} &= \frac{1}{2}(\sigma_1 + \sigma_2) - \frac{1}{2}(\sigma_1 - \sigma_2)\cos 2\theta \\ \sigma'_{xy} &= -\frac{1}{2}(\sigma_1 - \sigma_2)\sin 2\theta\end{aligned}} \tag{3.5.8}$$

Here, θ is measured *from* the principal directions, as illustrated in Fig. 3.5.5.

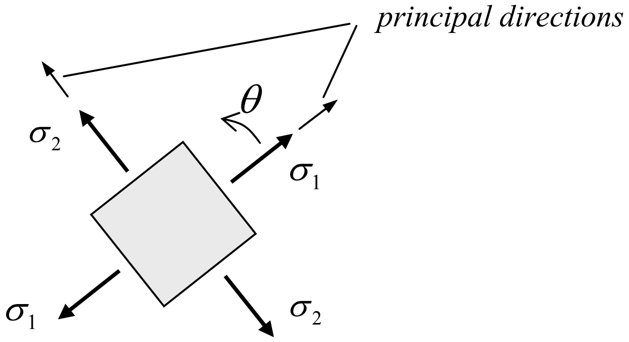


Figure 3.5.5: principal stresses and principal directions

The Third Principal Stress

Although plane stress is essentially a two-dimensional stress-state, it is important to keep in mind that any real material is three-dimensional. The stresses acting *on* the $x-y$ plane are the normal stress σ_{zz} and the shear stresses σ_{zx} and σ_{zy}, Fig. 3.5.6. These are all zero (in plane stress). It was discussed above how the principal stresses occur on planes of zero shear stress. Thus the σ_{zz} stress is also a principal stress. Technically speaking, there are always three principal stresses in three dimensions, and (at least) one of these will be zero in plane stress. This fact will be used below in the context of maximum shear stress.

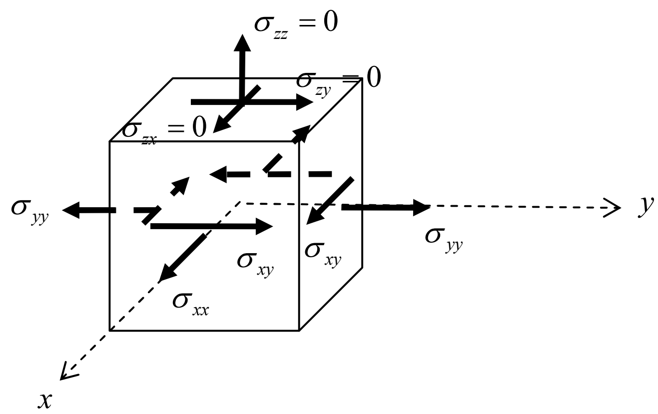


Figure 3.5.6: stresses acting on the $x-y$ plane

Maximum Shear Stress

Eqns. 3.5.8 can be used to derive an expression for the maximum shear stress. Differentiating the expression for shear stress with respect to θ, setting to zero and solving, shows that the maximum/minimum occurs at $\theta = \pm 45$, in which case

$$\sigma_{xy}\Big|_{\theta=+45} = -\frac{1}{2}(\sigma_1 - \sigma_2), \qquad \sigma_{xy}\Big|_{\theta=-45} = +\frac{1}{2}(\sigma_1 - \sigma_2)$$

or

$$\boxed{\max(\sigma_{xy}) = \frac{1}{2}|\sigma_1 - \sigma_2|} \qquad \textbf{Maximum Shear Stress} \qquad (3.5.9)$$

Thus the shear stress reaches a maximum on planes which are oriented at $\pm 45°$ to the principal planes, and the value of the shear stress acting on these planes is as given above. Note that the formula Eqn. 3.5.9 does not let one know in which *direction* the shear stresses are acting but this is not usually an important issue. Many materials respond in certain ways when the maximum shear stress reaches a critical value, and the actual direction of shear stress is unimportant. The direction of the maximum principal stress is, on the other hand, important – a material will in general respond differently according to whether the normal stress is compressive or tensile.

The normal stress acting on the planes of maximum shear stress can be obtained by substituting $\theta = \pm 45$ back into the formulae for normal stress in Eqn. 3.5.8, and one sees that

$$\sigma'_{xx} = \sigma'_{yy} = (\sigma_1 + \sigma_2)/2 \tag{3.5.10}$$

The results of this section are summarised in Fig. 3.5.7.

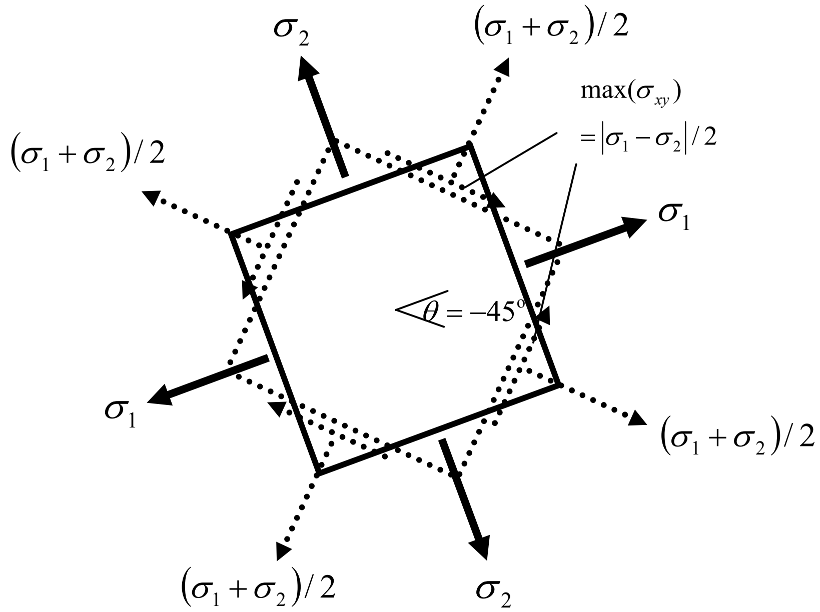
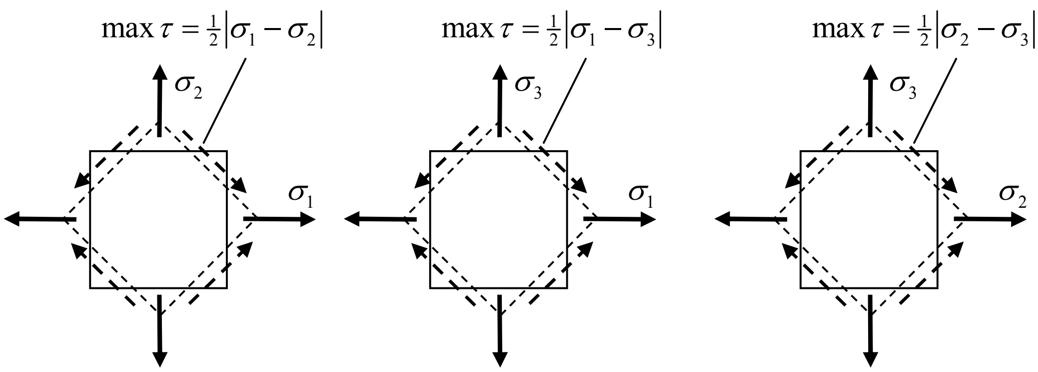


Figure 3.5.7: principal stresses and maximum shear stresses acting on the $x - y$ plane

The maximum shear stress in the $x - y$ plane was calculated above, Eqn. 3.5.9. This is not necessarily the maximum shear stress acting at the material particle. In general, it can be shown that the maximum shear stress is the maximum of the following three terms (see the Appendix to this section, §3.5.8):

$$\frac{1}{2}|\sigma_1 - \sigma_2|, \quad \frac{1}{2}|\sigma_1 - \sigma_3|, \quad \frac{1}{2}|\sigma_2 - \sigma_3|$$

The first term is the maximum shear stress in the $1-2$ plane, i.e. the plane containing the σ_1 and σ_2 stresses (and given by Eqn. 3.5.9). The second term is the maximum shear stress in the $1-3$ plane and the third term is the maximum shear stress in the $2-3$ plane. These are sketched in Fig. 3.5.8 below.



(a) 1-2 plane (a) 1-3 plane (a) 2-3 plane

Figure 3.5.8: principal stresses and maximum shear stresses

In the case of plane stress, $\sigma_3 = \sigma_{zz} = 0$, and the maximum shear stress will be

$$\max\left\{\frac{1}{2}|\sigma_1 - \sigma_2|, \quad \frac{1}{2}|\sigma_1|, \quad \frac{1}{2}|\sigma_2|\right\} \tag{3.5.11}$$

3.5.3 Stress Boundary Conditions

When solving problems, information is usually available on what is happening at the boundaries of materials. This information is called the **boundary conditions**. Information is usually not available on what is happening in the interior of the material – information there is obtained by solving the equations of mechanics.

A number of different conditions can be known at a boundary, for example it might be known that a certain part of the boundary is fixed so that the displacements there are zero. This is known as a **displacement boundary condition**. On the other hand the stresses over a certain part of the material boundary might be known. These are known as **stress boundary conditions** – this case will be examined here.

General Stress Boundary Conditions

It has been seen already that, when one material contacts a second material, a force, or distribution of stress arises. This force F will have arbitrary direction, Fig. 3.5.9a, and can be decomposed into the sum of a normal stress distribution σ_N and a shear distribution σ_S, Fig. 3.5.9b. One can introduce a coordinate system to describe the applied stresses, for example the $x-y$ axes shown in Fig. 3.5.9c (the axes are most conveniently defined to be normal and tangential to the boundary).

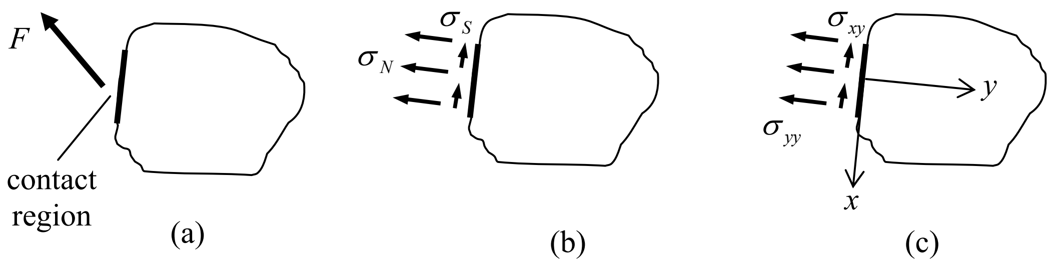


Figure 3.5.9: Stress boundary conditions; (a) force acting on material due to contact with a second material, (b) the resulting normal and shear stress distributions, (c) applied stresses as stress components in a given coordinate system

Figure 3.5.10 shows the same component as Fig. 3.5.9. Shown in detail is a small material element at the boundary. From equilibrium of the element, stresses σ_{xy}, σ_{yy}, equal to the applied stresses, must be acting inside the material, Fig. 3.5.10a. Note that the **tangential stresses**, which are the σ_{xx} stresses in this example, can take on any value and the element will still be in equilibrium with the applied stresses, Fig. 3.5.10b.

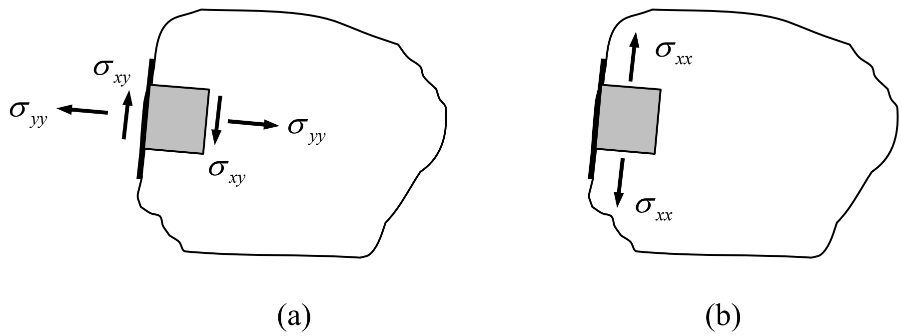


Figure 3.5.10: Stresses acting on a material element at the boundary, (a) normal and shear stresses, (b) tangential stresses

Thus, if the applied stresses are *known*, then so also are the normal and shear stresses acting at the boundary of the material.

Stress Boundary Conditions at a Free Surface

A free surface is a surface that has "nothing" on one side and so there is nothing to provide reaction forces. Thus there must also be no normal or shear stress on the other side (the inside).

This leads to the following, Fig. 3.5.11:

Stress boundary conditions at a free surface:
the normal and shear stress at a free surface are zero

This simple fact is used again and again to solve practical problems.

Again, the stresses acting normal to any other plane at the surface do not have to be zero – they can be balanced as, for example, the tangential stresses σ_T and the stress $\overline{\sigma}$ in Fig. 3.5.11.

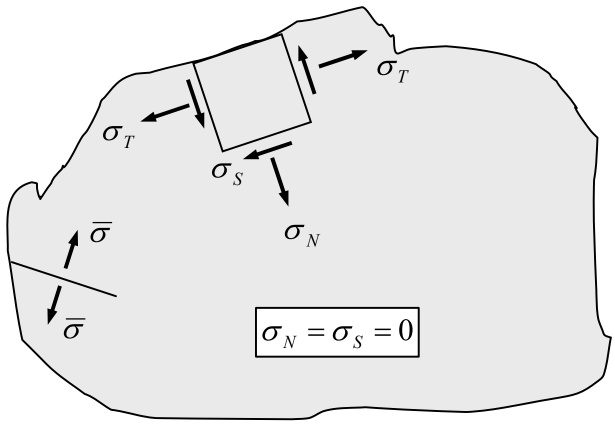


Figure 3.5.11: A free surface - the normal and shear stresses there are zero

Atmospheric Pressure

There *is* something acting on the outside "free" surfaces of materials – the atmospheric pressure. This is a type of stress which is **hydrostatic**, that is, it acts normal at all points, as shown in Fig. 3.5.12. Also, it does not vary much. This pressure is present when one characterises a material, that is, when its material properties are determined from tests and so on, for example, its Young's Modulus (see Chapter 5). The atmospheric pressure is therefore a datum – stresses are really measured relative to this value, and so the atmospheric pressure is ignored.

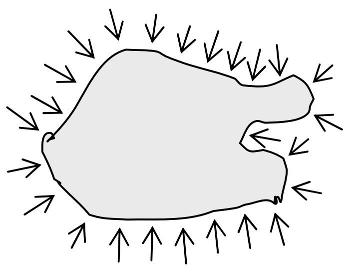


Figure 3.5.12: a material subjected to atmospheric pressure

3.5.4 Thin Components

Consider a thin component as shown in Fig. 3.5.13. With the coordinate axes aligned as shown, and with the large face free of loading, one has $\sigma_{zx} = \sigma_{zy} = \sigma_{zz} = 0$. Strictly speaking, these stresses are zero *only* at the free surfaces of the material but, because it is thin, these stresses should not vary much from zero within. Taking the "*z*" stresses to be identically zero throughout the material, the component is in a state of plane stress[1]. On the other hand, were the sheet not so thin, the stress components that were zero at the free-surfaces might well deviate significantly from zero deep within the material.

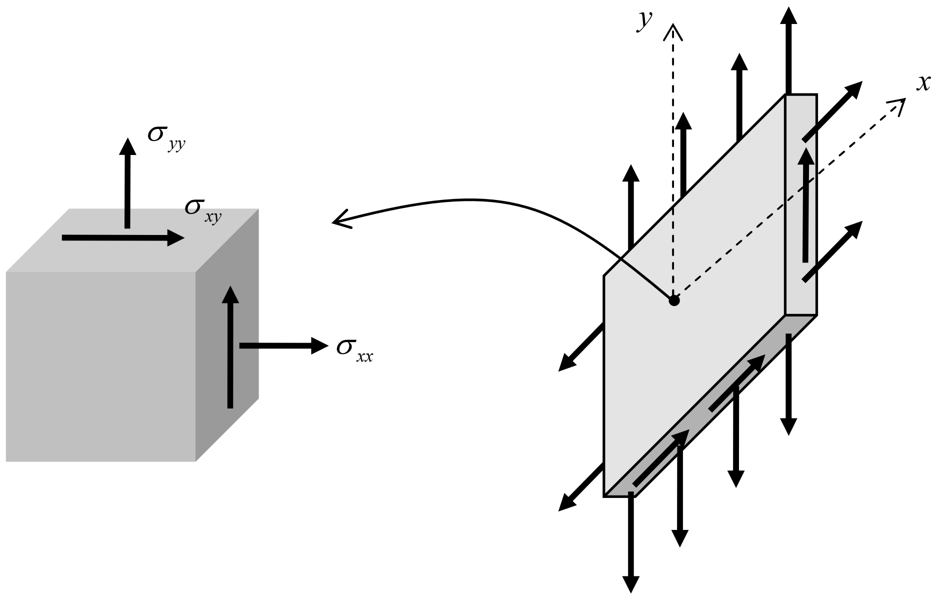
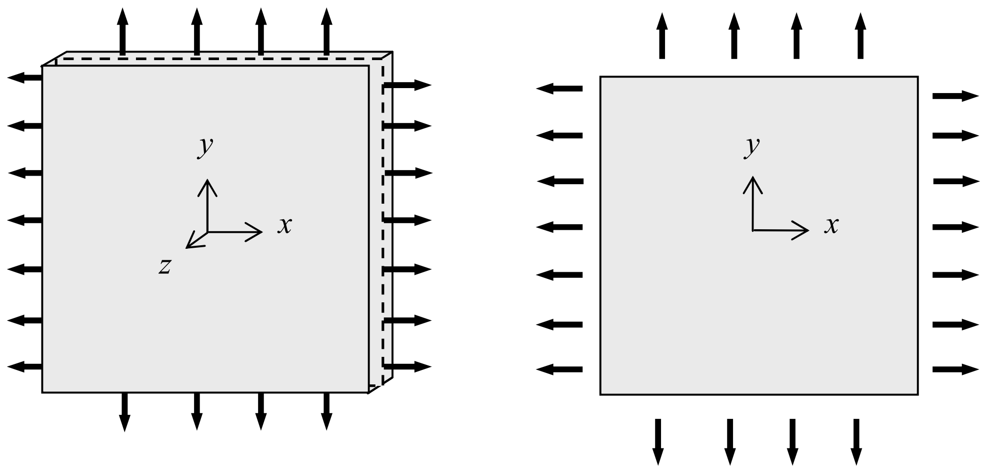


Figure 3.5.13: a thin material loaded in-plane, leading to a state of plane stress

When analysing plane stress states, only one cross section of the material need be considered. This is illustrated in Fig. 3.5.14.

[1] it will be shown in Book II that, when the applied stresses σ_{xx}, σ_{yy}, σ_{xy} vary only *linearly* over the thickness of the component, the stresses σ_{zz}, σ_{zx}, σ_{zy} are exactly zero throughout the component, otherwise they are only approximately zero



3 – dimensional material

2 – dimensional cross-section of material

Figure 3.5.14: one two-dimensional cross-section of material

Note that, although the stress normal to the plane, σ_{zz}, is zero, the three dimensional sheet of material *is* deforming in this direction – it will obviously be getting thinner under the tensile loading shown in Fig. 3.5.14.

Note that plane stress arises in *all* thin materials (loaded in –plane), no matter what they are made of.

3.5.5 Mohr's Circle

Otto Mohr devised a way of describing the state of stress at a point using a single diagram, called the **Mohr's circle**.

To construct the Mohr circle, first introduce the **stress coordinates** (σ, τ), Fig. 3.5.15; the abscissae (horizontal) are the normal stresses σ and the ordinates (vertical) are the shear stresses τ. On the horizontal axis, locate the principal stresses σ_1, σ_2, with $\sigma_1 > \sigma_2$. Next, draw a circle, centred at the average principal stress $(\sigma, \tau) = ((\sigma_1 + \sigma_2)/2, 0)$, having radius $(\sigma_1 - \sigma_2)/2$.

The normal and shear stresses acting on a single plane are represented by a single point on the Mohr circle. The normal and shear stresses acting on two perpendicular planes are represented by two points, one at *each end of a diameter* on the Mohr circle. Two such diameters are shown in the figure. The first is horizontal. Here, the stresses acting on two perpendicular planes are $(\sigma, \tau) = (\sigma_1, 0)$ and $(\sigma, \tau) = (\sigma_2, 0)$ and so this diameter represents the principal planes/stresses.

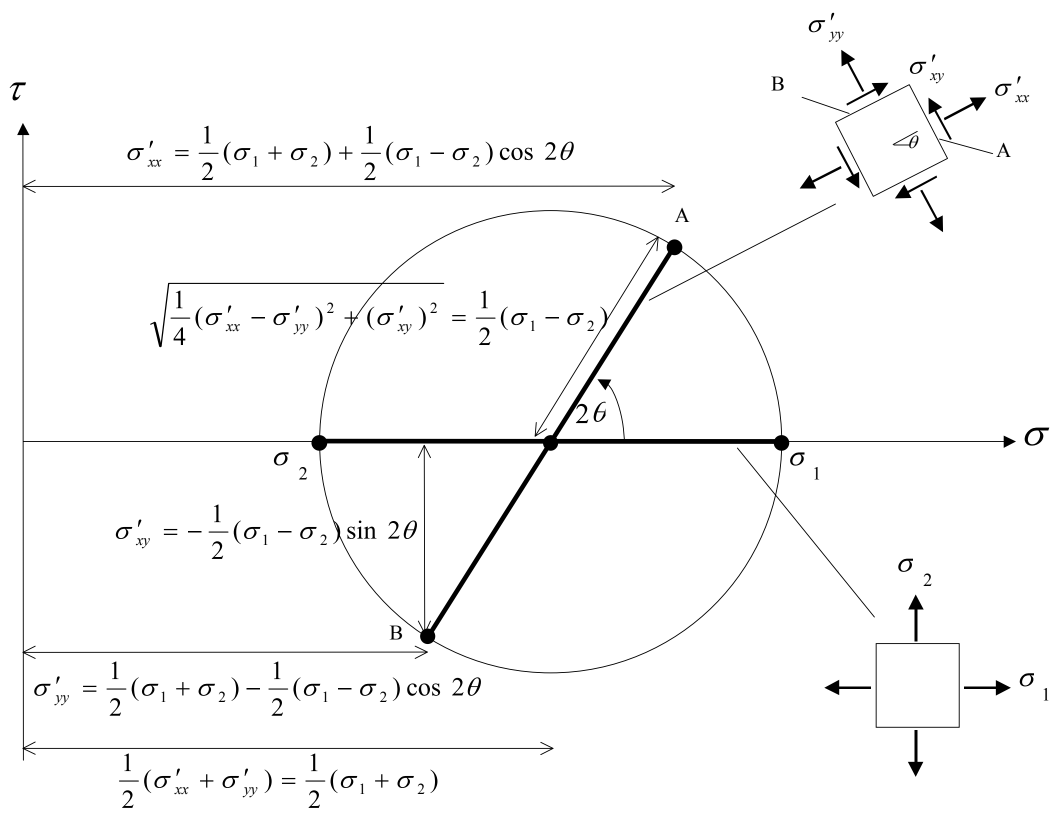


Figure 3.5.15: Mohr's Circle

The stresses on planes rotated by an amount θ from the principal planes are given by Eqn. 3.5.8. Using elementary trigonometry, these stresses are represented by the points A and B in Fig. 3.5.15. Note that a rotation of θ in the physical plane corresponds to a rotation of 2θ in the Mohr diagram.

Note also that the conventional labeling of shear stress has to be altered when using the Mohr diagram. On the Mohr circle, a shear stress is positive if it yields a clockwise moment about the centre of the element, and is "negative" when it yields a negative moment. For example, at point A the shear stress is "positive" ($\tau > 0$), which means the direction of shear on face A of the element is actually opposite to that shown. This agrees with the formula $\sigma'_{xy} = -\frac{1}{2}(\sigma_1 - \sigma_2)\sin 2\theta$, which is less than zero for $\sigma_1 > \sigma_2$ and $\theta \le 90^\circ$. At point B the shear stress is "negative" ($\tau < 0$), which again agrees with formula.

3.5.6 Stress Boundary Conditions (continued)

Consider now in more detail a surface between two different materials, Fig. 3.5.16. One says that the normal and shear stresses are **continuous** across the surface, as illustrated.

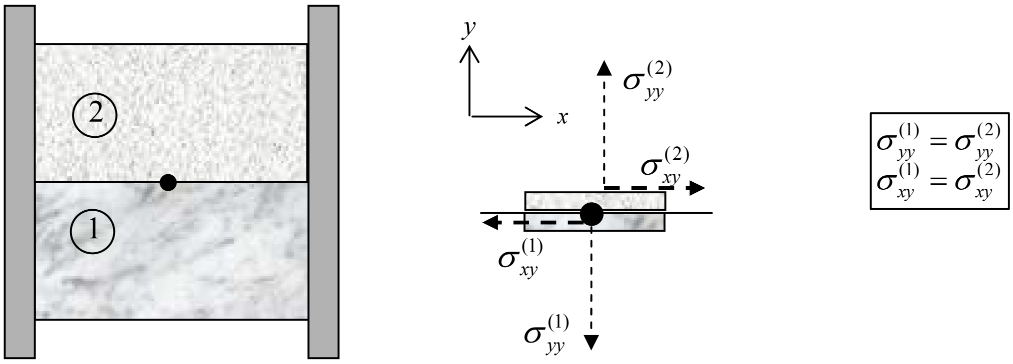


Figure 3.5.16: normal and shear stress continuous across an interface between two different materials, material '1' and material '2'

Note also that, since the shear stress σ_{xy} is the same on both sides of the surface, the shear stresses acting on both sides of a perpendicular plane passing *through* the interface between the materials, by the symmetry of stress, must also be the same, Fig. 3.5.17a.

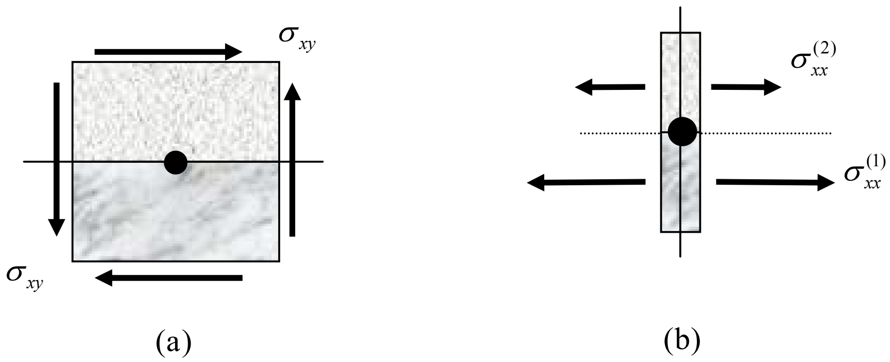


Figure 3.5.17: stresses at an interface; (a) shear stresses continuous across the interface, (b) tangential stresses not necessarily continuous

However, again, the tangential stresses, those acting parallel to the interface, do *not* have to be equal. For example, shown in Fig. 3.5.17b are the tangential stresses acting in the upper material, $\sigma_{xx}^{(2)}$ - they balance no matter what the magnitude of the stresses $\sigma_{xx}^{(1)}$.

Description of Boundary Conditions

The following example brings together the notions of stress boundary conditions, stress components, equilibrium and equivalent forces.

Example

Consider the plate shown in Fig. 3.5.18. It is of width $2a$, height *b* and depth *t*. It is subjected to a tensile stress *r*, pressure *p* and shear stresses *s*. The applied stresses are uniform through the thickness of the plate. It is welded to a rigid base.

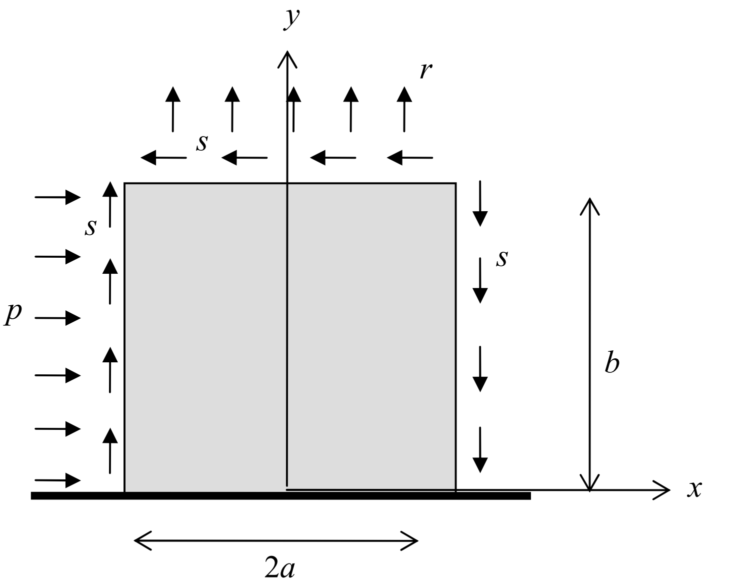


Figure 3.5.18: a plate subjected to stress distributions

Using the $x-y$ axes shown, the stress boundary conditions can be expressed as:

Left-hand surface: $\begin{cases} \sigma_{xx}(-a,y) = -p \\ \sigma_{xy}(-a,y) = -s \end{cases}, \quad 0 < y < b$

Top surface: $\begin{cases} \sigma_{yy}(x,b) = +r \\ \sigma_{xy}(x,b) = -s \end{cases}, \quad -a < x < +a$

Right-hand surface: $\begin{cases} \sigma_{xx}(+a,y) = 0 \\ \sigma_{xy}(+a,y) = -s \end{cases}, \quad 0 < y < b$

Note carefully the description of the normal and shear stresses over each side and the signs of the stress components.

The stresses at the lower edge are unknown (there is a displacement boundary condition there: zero displacement). They will in general not be uniform. Using the given $x-y$ axes, these unknown reaction stresses, exerted by the base on the plate, are (see Fig 3.5.19)

Lower surface: $\begin{cases}\sigma_{yy}(x,0) \\ \sigma_{xy}(x,0)\end{cases}, \qquad -a < x < +a$

Note the directions of the arrows in Fig. 3.5.19, they have been drawn in the direction of positive $\sigma_{yy}(x,0)$, $\sigma_{xy}(x,0)$.

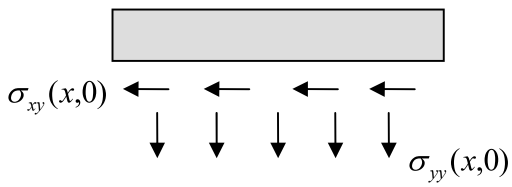


Figure 3.5.19: unknown reaction stresses acting on the lower edge

For force equilibrium of the complete plate, consider the free-body diagram 3.5.20; shown are the resultant forces of the stress distributions. Force equilibrium requires that

$$\sum F_x = bpt - 2ast - t\int_{-a}^{+a}\sigma_{xy}(x,0)dx = 0$$
$$\sum F_y = 2art - t\int_{-a}^{+a}\sigma_{yy}(x,0)dx = 0$$

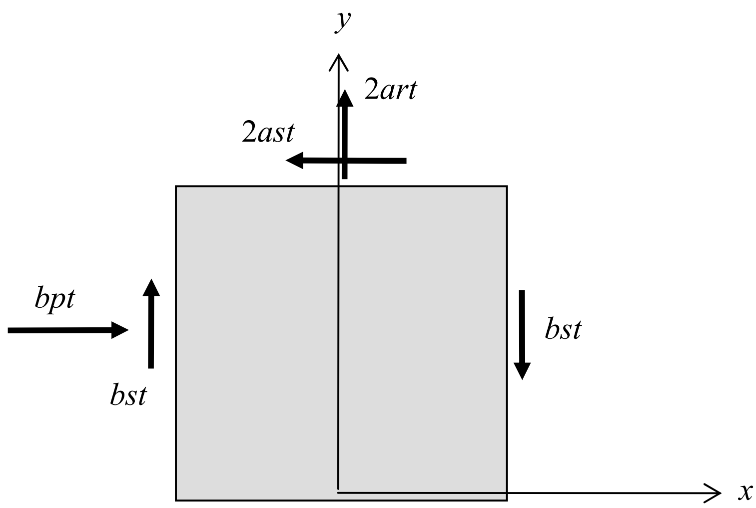


Figure 3.5.20: a free-body diagram of the plate in Fig. 3.5.18 showing the known resultant forces (forces on the lower boundary are not shown)

For moment equilibrium, consider the moments about, for example, the lower left-hand corner. One has

$$\sum M_0 = -bpt(b/2) + 2ast(b) + 2art(a) - bst(2a) - t\int_{-a}^{+a} \sigma_{yy}(x,0) \times (a+x)dx = 0$$

If one had taken moments about the top-left corner, the equation would read

$$\sum M_0 = +bpt(b/2) + 2art(a) - bst(2a)$$
$$- t\int_{-a}^{+a} \sigma_{xy}(x,0) \times b\,dx - t\int_{-a}^{+a} \sigma_{yy}(x,0) \times (a+x)dx = 0$$

■

Source:http://homepages.engineering.auckland.ac.nz/~pkel015/SolidMechanicsBooks/Part_I/BookSM_Part_I/03_Stress/03_Stress_05_Plane_Stress.pdf