Orbit Transfers and Interplanetary Trajectories

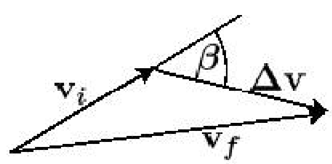
In this lecture, we will consider how to transfer from one orbit, to another or to construct an interplanetary trajectory. One of the assumptions that we shall make is that the velocity changes of the spacecraft, due to the propulsive effects, occur instantaneously. Although it obviously takes some time for the spacecraft to accelerate to the velocity of the new orbit, this assumption is reasonable when the burn time of the rocket is much smaller than the period of the orbit. In such cases, the Δv required to do the maneuver is simply the difference between the velocity of the final orbit minus the velocity of the initial orbit.

When the initial and final orbits intersect, the transfer can be accomplished with a single impulse. For more general cases, multiple impulses and intermediate transfer orbits may be required.

Given initial and final orbits, the objective is generally to perform the transfer with a minimum Δv. In some situations, however, the time needed to complete the transfer may also be an important consideration.

Most orbit transfers will require a change in the orbit's total specific energy, E. Let us consider the change in total energy obtained by an instantaneous impulse Δv. If $\boldsymbol{v}_i$ is the initial velocity, the final velocity, $\boldsymbol{v}_f$, will simply be,

$$\boldsymbol{v}_f = \boldsymbol{v}_i + \Delta \boldsymbol{v} \ .$$



If we now look at the magnitude of these vectors, we have,

$$v_f^2 = v_i^2 + \Delta v^2 + 2 v_i \Delta v \cos\beta,$$

where β is the angle between $\boldsymbol{v}_i$ and Δv. The energy change will be

$$\Delta E = \frac{1}{2}\Delta v^2 + v_i \Delta v \cos\beta \ .$$

From this expression, we conclude that, for a given Δv, the change in energy will be largest when:

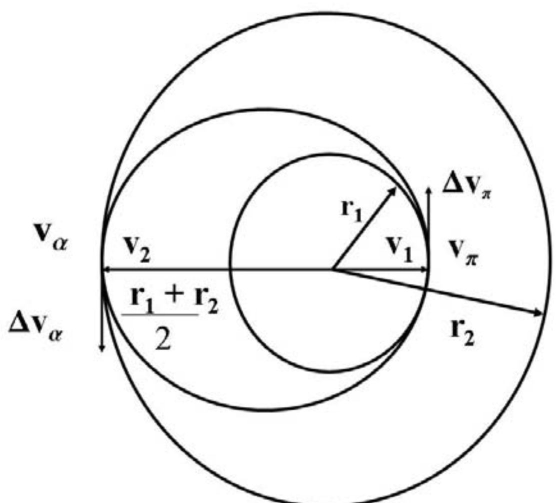
- $\boldsymbol{v}_i$ and $\Delta \boldsymbol{v}$ are co-linear ($\beta = 0$), and,

- v_i is maximum.

For example, to transfer a satellite on an elliptical orbit to an escape trajectory, the most energy efficient impulse would be co-linear with the velocity and applied at the instant when the satellite is at the elliptical orbit's perigee, since at that point, the velocity is maximum. Of course, for many required maneuvers, the applied impulses are such that they cannot satisfy one or both of the above conditions. For instance, firing at the perigee in the previous example may cause the satellite to escape in a particular direction which may not be the required one.

Hohmann Transfer

A Hohmann Transfer is a two-impulse elliptical transfer between two co-planar circular orbits. The transfer itself consists of an elliptical orbit with a perigee at the inner orbit and an apogee at the outer orbit. The fundamental assumption behind the Hohmann transfer, is that there is only one body which exerts a gravitational force on the body of interest, such as a satellite. This is a good model for transferring an earth-based satellite from a low orbit to say a geosynchronous orbit. Inherent in the model is that there is no additional body sharing the orbit which could induce a gravitational attraction on the body of interest. Thus, as we shall discuss, the Hohmann transfer is a good model for the "outer" trajectory of an earth-Mars transfer, but we must pay some attention to "escaping" the earth's gravitational field before we're on our way.



Outer solution: Hohmann transfer

It turns out that this transfer is usually optimal, as it requires the minimum $\Delta v_T = |\Delta v_\pi| + |\Delta v_\alpha|$ to perform a transfer between two circular orbits. The exception for which Hohmann transfers are not optimal is for very large ratios of r_2/r_1, as discussed below.

The transfer orbit has a semi-major axis, a, which is

$$a = \frac{r_1 + r_2}{2} \ .$$

Hence, the energy of the transfer orbit is greater than the energy of the inner orbit ($a = r_1$), and smaller than the energy of the outer orbit ($a = r_2$). The velocities of the transfer orbit at perigee and apogee are given, from the conservation of energy equation, as

$$v_\pi^2 = \mu\left(\frac{2}{r_1} - \frac{2}{r_1+r_2}\right) \quad (1)$$

$$v_\alpha^2 = \mu\left(\frac{2}{r_2} - \frac{2}{r_1+r_2}\right) . \quad (2)$$

The velocities of the circular orbits are $v_{c1} = \sqrt{\mu/r_1}$ and $v_{c2} = \sqrt{\mu/r_2}$. Hence, the required impulses at perigee and apogee are,

$$\Delta v_\pi = v_\pi - v_{c1} = \sqrt{\frac{\mu}{r_1}}\left(\sqrt{\frac{2r_2}{r_1+r_2}} - 1\right)$$

$$\Delta v_\alpha = v_{c2} - v_\alpha = \sqrt{\frac{\mu}{r_2}}\left(1 - \sqrt{\frac{2r_1}{r_1+r_2}}\right) .$$

If the initial orbit has a radius larger than the final orbit, the same strategy can be followed but in this case, negative impulses will be required, first at apogee and then at perigee, to decelerate the satellite.

Example ***Hohmann transfer [1]***

A communication satellite was carried by the Space Shuttle into low earth orbit (LEO) at an altitude of 322 km and is to be transferred to a geostationary orbit (GEO) at 35,860 km using a Hohmann transfer. The characteristics of the transfer ellipse and the total Δv required, Δv_T, can be determined as follows:
For the inner orbit, we have,

$$r_1 = R + d_1 = 6.378 \times 10^6 + 322 \times 10^3 = 6.70 \times 10^6 \,\mathrm{m}$$

$$v_{c1} = \sqrt{\frac{\mu}{r_1}} = \sqrt{\frac{gR^2}{r_1}} = 7713 \,\mathrm{m/s}$$

$$E_1 = -\frac{\mu}{2r_1} = -\frac{gR^2}{2r_1} = -2.975 \times 10^7 \,\mathrm{m^2/s^2}\ (\mathrm{J/kg}).$$

Similarly, for the outer circular orbit,

$$r_2 = R + d_2 = 42.24 \times 10^6 \,\mathrm{m}$$

$$v_{c2} = 3072 \,\mathrm{m/s}$$

$$E_2 = -4.718 \times 10^6 \,\mathrm{m^2/s^2}\ (\mathrm{J/kg}).$$

For the transfer trajectory,

$$2a = r_1 + r_2 = 48.94 \times 10^6 \,\mathrm{m}$$

$$E = -\frac{\mu}{2a} = -8.144 \times 10^6 \,\mathrm{m^2/s^2}\ (\mathrm{J/kg}),$$

which shows that $E_1 < E < E_2$. The velocity of the elliptical transfer orbit at the perigee and apogee can be determined from equations 3 and 4, as,

$$\begin{aligned} v_\pi &= 10,130\,\mathrm{m/s}\ , \\ v_\alpha &= 1,067\,\mathrm{m/s}\ . \end{aligned}$$

Since the velocity at the perigee is orthogonal to the position vector, the specific angular momentum of the transfer orbit is,

$$h = r_1 v_\pi = 6.787 \times 10^{10}\,\mathrm{m^2/s}\ ,$$

and the eccentricity can be determined as,

$$e = \sqrt{1 + \frac{2Eh^2}{\mu^2}} = 0.7265\ .$$

Finally, the impulses required are,

$$\begin{aligned} \Delta v_\pi &= v_\pi - v_{c1} = 10,130 - 7713 = 2414\,\mathrm{m/s} \\ \Delta v_\alpha &= v_{c2} - v_\alpha = 3072 - 1607 = 1465\,\mathrm{m/s}, \end{aligned}$$

The sign of the Δv's indicates the direction of thrusting (whether the energy is to be increased or decreased) and the total Δv is the sum of the magnitudes. Thus,

$$\Delta v_T = |\Delta v_\pi| + |\Delta v_\alpha| = 2417 + 1465 = 3882\,\mathrm{m/s}.$$

Since the transfer trajectory is one half of an ellipse, the time of flight (TOF) is simply half of the period,

$$TOF = \pi \sqrt{\frac{(24.47 \times 10^6)^3}{3.986 \times 10^{14}}} = 19,050\,\mathrm{s} = 5.29\,\mathrm{h}\ .$$

In order to illustrate the optimal nature of the Hohmann transfer, we consider now an alternative transfer in which we arbitrarily double the value of the semi-major axis of the Hohmann transfer ellipse, and find the characteristics and Δv_T of the resulting fast transfer. The semi-major axis of the transfer ellipse will be $2a = 98 \times 10^6$ m, and $E = -\mu/(2a) = -4.067 \times 10^6$ $\mathrm{m^2/s^2}$ (J/kg). The velocity of the transfer orbit at departure will be

$$v_\pi = \sqrt{2\left(-4.067 \times 10^6 + \frac{3.986 \times 10^{14}}{6.70 \times 10^6}\right)} = 10.530\,\mathrm{m/s},$$

and,

$$\Delta v_\pi = 10,530 - 7713 = 2817\,\mathrm{m/s}\ .$$

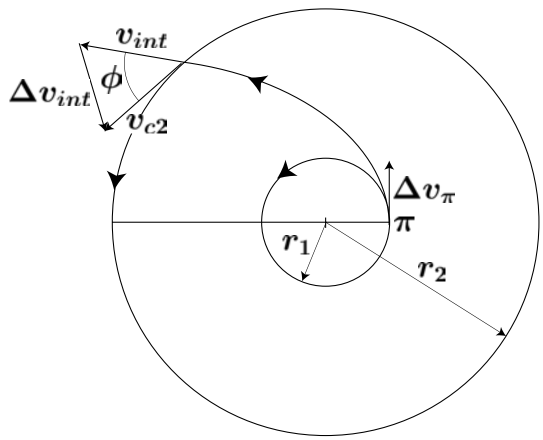
The specific angular momentum is,

$$h = 10,530 \times (6.70 \times 10^6) = 7.055 \times 10^{10}\,\mathrm{m^2/s},$$

and the orbit's eccentricity, e, is 0.863.

We can now calculate, from the energy conservation equation, the velocity of the transfer orbit at the point of interception with the outer orbit, v_{int},

$$v_{int} = \sqrt{2\left(-4.067 \times 10^6 + \frac{3.986 \times 10^{14}}{42.24 \times 10^6}\right)} = 3277\,\text{m/s} \ .$$



Since the angular momentum, h, is conserved, we can determine the component of $\boldsymbol{v}_{int}$ in the circumferential direction

$$(\boldsymbol{v}_{int})_\theta = \frac{h}{r_2} = 1670\,\text{m/s}$$

and the elevation angle, ϕ, is thus,

$$\phi = \cos^{-1}\frac{(\boldsymbol{v}_{int})_\theta}{v_{int}} = 59.36^o$$

Finally, from geometrical considerations,

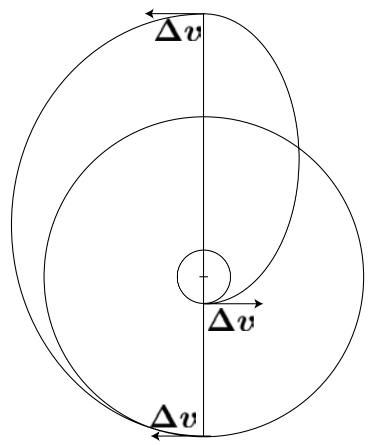
$$\Delta v_{int}^2 = v_{c2}^2 + v_{int}^2 - 2v_{c2}v_{int}\cos\phi,$$

which yields $\Delta v_{int} = 3142$ m/s, so that

$$\Delta v_T = 2817 + 3142 = 5959\,\text{m/s}.$$

Comparing to the value of the Hohmann transfer Δv_T of 3875 m/s, we see that the fast transfer requires a Δv_T which is 54% higher. The analysis of elliptical trajectories which intersect the circular orbits at an angle is referred to as Lambert's problem. These trajectories can have faster transit times but at a greater cost in energy.

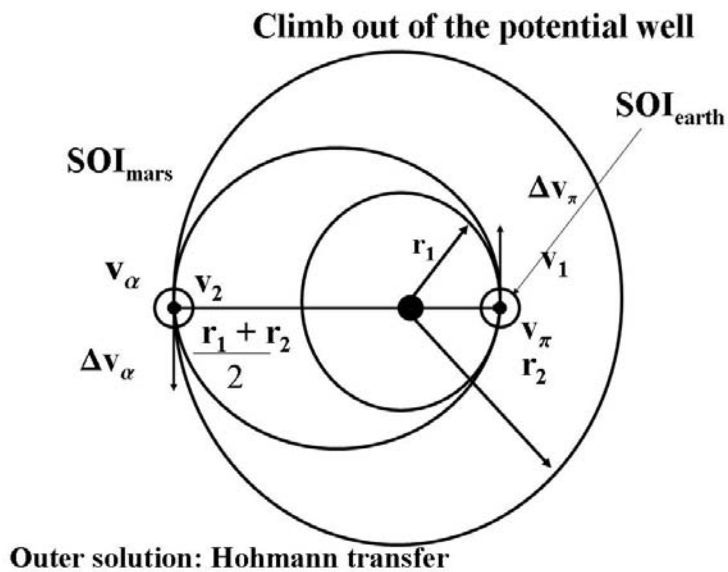
It can be shown that when the separation between the inner and outer orbits is very large ($r_2 > 11.9 r_1$) (a situation which rarely occurs), a three impulse transfer comprising of two ellipses can be more energy efficient than a two-impulse Hohmann transfer. This transfer is illustrated in the picture below. Notice that the distance from the origin at which the two transfer ellipses intersect is a free parameter, which can be determined to minimize the total Δv. Notice also that the final impulse is a Δv which opposes direction of motion, in order to decelerate from the large energy ellipse to the final circular orbit. Although this transfer may be more energy efficient relative to the two-impulse Hohmann transfer, it often involves much larger travel times. Try it!



Interplanetary Transfers

The ideas of Hohmann transfer can be applied to interplanetary transfers with some modification. The Hohmann transfer for satellite orbits assumes the satellite is in a circular orbit about a central body and desires to transfer to another circular and coplanar orbit about the central body. It also assumes that no other gravitational influence is nearby. When more than one planet is involved, such as a satellite in earth orbit which desires to transfer via a Hohmann orbit about the sun to an orbit about another planet, such as done in a mission to Mars. In this case, the problem is no longer a two-body problem.

Nevertheless, it is common (at least to get a good approximation) to decompose the problem into a series of two body problems. Consider, for example, an interplanetary transfer in the solar system. For each planet we define the sphere of influence (SOI). Essentially, this is the region where the gravitational attraction due to the planet is larger than that of the sun. In order to be on our way to the destination planet, we must climb out of the potential well of the originating planet. We will use a hyperbolic ”escape” orbit to accomplish this. Alternatively, we could do a direct calculation including the position of all of the bodies: the sun, earth and Mars. However, because the time scale and length scales are so different for the different phases of the mission, it requires special attention to the details of the numerical method to attain good accuracy. The method of patched conics is a good place to start our analysis.



The mission is broken into phases that are connected by patches where each patch is the solution of a two body problem. This is called the patched conic approach. Consider, for instance, a mission to Mars. The first phase will consist of a geocentric hyperbola as the spacecraft escapes from earth SOI, attaining a velocity v_1 in a direction θ beyond the earth's SOI. The second phase would start at the edge of the earth's SOI, and would be an elliptical trajectory around the sun while the spacecraft travels to Mars. This orbit could be part of a Hohmann transfer sequence; in this case, v_1 would be the Hohmann transfer velocity after the ΔV has been applied. The third phase would start at the edge of Mars' SOI, and would be a hyperbolic approach capture trajectory with the gravitational field of Mars as the attracting force. This third phase can be thought of as a combination of a Hohmann transfer and a hyperbolic capture by the planet.

The time scales and length scales for the various phases of the mission are quite different. The time for a transfer to another planet is measured in months or years; the time scale for escape for a planet is measured in days or hours. The length scale for planetary trajectories is measured in AU units where AU is the distance from the earth to the sun; the length scale for hyperbolic escape from a planet is measure in distances typical of planetary radii and orbits. This can be a challenge for an orbit calculation program since the step size must change dramatically near a planet. We will examine this problem analytically using the method of matched conics to get an approximate result. This approach does not work well for trajectories from earth to moon since the moon is in the SOI of the earth.

In turns out that a Hohmann transfer orbit with hyperbolic escape and capture trajectories can be shown to be the minimum energy trajectory for a planetary mission just as the Hohmann transfer itself is an optimum solution for a specified satellite change of orbit. Of course a successful mission requires that the time of launch be selected so that the desired destination planet is at the destination when the spacecraft arrives. We assume that the proper launch time has been selected.

Hyperbolic Escape

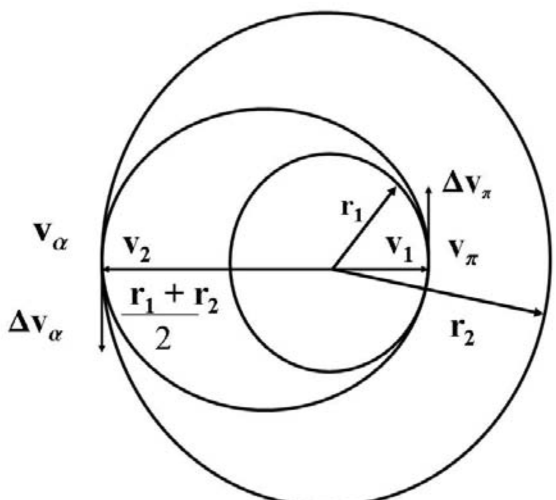
We now consider a hyperbolic escape from a planetary orbit of a planet in orbit about the sun into an elliptical orbit about the sun determined by a Hohmann transfer to the desired planet combined with a hyperbolic capture into a planetary orbit about the destination planet in orbit about the sun for a complete interplanetary transfer. This sequence is appropriate for travel to an outer planet such as Mars.

We first consider the Hohmann transfer from a circular orbit about the sun coincident with the original planetary orbit to a circular coplanar orbit coincident with the destination planet's orbit about the sun. We neglect the gravitational fields of the two planets in this calculations. From Equations (1-4) we have v_π and v_α for the elliptical connecting orbit and

$$v_\pi^2 = \mu\left(\frac{2}{r_1} - \frac{2}{r_1 + r_2}\right) \tag{3}$$

$$v_\alpha^2 = \mu\left(\frac{2}{r_2} - \frac{2}{r_1 + r_2}\right) \tag{4}$$

and for the circular planetary orbits at r_1 and r_2 (Note that we have used the symbol μ without designation. Obviously, the choice of μ in a particular calculation depends upon the celestial body supporting the orbit.) $v_1 = \sqrt{\mu/r_1}$ and $v_2 = \sqrt{\mu/r_2}$ for the initial and final circular orbits about the sun.



Outer solution: Hohmann transfer

This determines the Δv_π and Δv_α for the Hohmann interplanetary transfer. It also determines the initial and final conditions for a hyperbolic escape trajectory.

Consider conditions at the departure planet, denoted by subscript 1. The planet moves with the circular orbit velocity v_1. The hyperbolic trajectory is defined in a coordinate system relative to the moving planet, and the escape trajectory must end with the velocity appropriate for the Hohmann transfer ellipse in inertial space, v_π or $v_\pi - v_1$ relative to the planet. The hyperbolic escape velocity v_∞ then equals exactly the Δv_π for a traditional Hohmann transfer. The only remaining unknown is the condition of the spacecraft when it decides to escape.

The geometry of a hyperbolic escape comes directly from the general geometry of a hyperbolic orbit. In this case we desired to depart on a vertical trajectory ending with a velocity $v_\infty = v_\pi - v_1$. If the spacecraft is in orbit around the earth at a velocity v_{orbit} and a radius r_1, this requires a $\Delta v_{escape} = v_{1p} - v_{orbit}$ where v_{1p} is the velocity of the hyperbolic escape orbit at the periapsis $r_p = r_1$.
Since energy is constant,

$$\frac{v_{1p}^2}{2} - \frac{\mu}{r_{1p}} = \frac{v_{1\infty}^2}{2}. \tag{5}$$

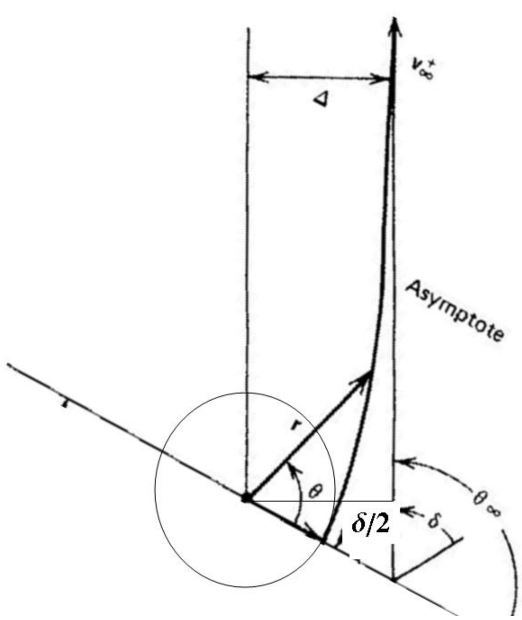
The parameters of the hyperbolic escape orbit follow from these conditions. The eccentricity is given by

$$\epsilon_1 = 1 + \frac{r_{1p} v_{1\infty}^2}{\mu} \tag{6}$$

and the angular change from periapsis to infinity, $\delta_1/2$, is given by

$$\delta_1/2 = \sin^{-1} \frac{1}{\epsilon_1} \tag{7}$$

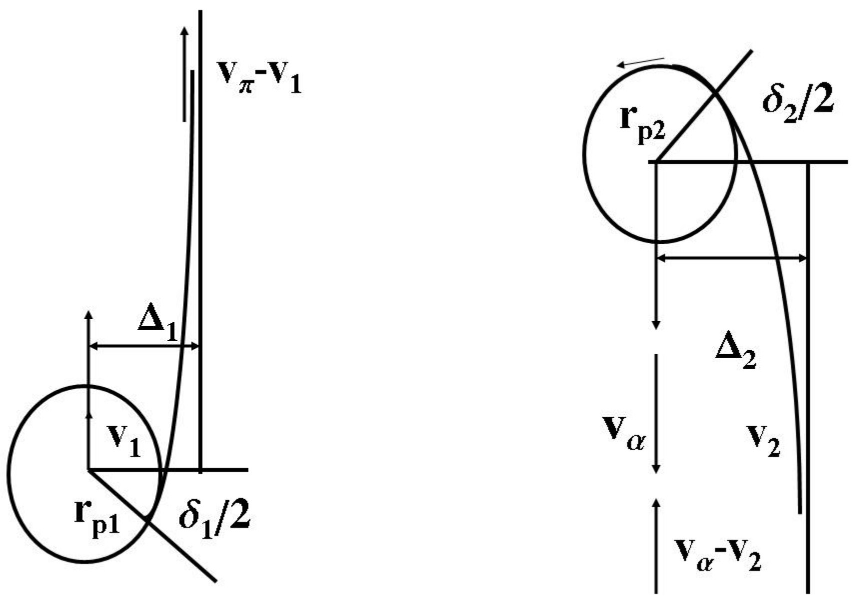
For the orbit geometry sketched below, this means that the launch from a circular orbit into a hyperbolic orbit with a final vertical direction as sketched occurs at an angle $-\delta_1/2$ as shown in the figure.



The trajectory shown is a counterclockwise escape trajectory. A clockwise escape trajectory is also possible but, because of the counterclockwise rotation of the earth, counterclockwise is preferred. One might worry about the sidewise displacement Δ_1 of the trajectory from the centerline of the planet orbit. However, this distance is very small compared to the interplanetary distances defined by the distance to the sun and is typical of the small errors inherent in the method of patched conics. Therefore, the hyperbolic escape trajectory has been defined as shown below. This enables the determination of the Δv_1 for this portion of the mission. The hyperbolic escape trajectory is shown below.

Hyperbolic Capture

Also shown below is the hyperbolic capture trajectory at the final planetary destination. Both these trajectories are shown for the case of travel to an outer planet such as Mars. For travel to an inner planet such as Venus, the roles of escape and capture would be reversed.



Hyperbolic escape **Hyperbolic capture**

We now consider the hyperbolic capture trajectory. From the Hohmann transfer calculation, we obtain the result that an additional Δv in the flight direction is required to circularize the orbit of the space craft into the destination planetary orbit, subscript 2. This means that the velocity v_α of the elliptical Hohmann transfer orbit is less that the velocity of the planet in its circular orbit. Thus the planet will overtake the spacecraft resulting in a hyperbolic capture orbit. The velocity $v_{2\infty}$ as seen by the planet is $v_\alpha - v_2$ and is directed towards the planet; this quantity plays the role of $v_{2\infty}$ in the hyperbolic capture orbit. We desire to capture the planet into an orbit of radius r_{2p} and inquire what conditions are require to achieve this. This will determine Δ_2 and v_{2p}. From conservation of energy, we have $v_{2p}^2/2 - \mu/r_{2p} = v_{2\infty}^2$; $\epsilon_2 = 1 + r_{2p}v_{2\infty}^2/\mu$; and $\sin\delta_2/2 = 1/\epsilon_2$. Of course some mid-course correction may be required to set Δ_2 to achieve this capture. This will be ignored.

The final step in the interplanetary mission would be to insert the spacecraft into a circular orbit of radius r_{2p}. We have chosen to capture the spacecraft in a planetary orbit proceeding in the direction of the planet's rotation about its axis. If we only intend to stay in orbit, this doesn't matter. However, if we intend to descend and later return to orbit, these maneuvers should be designed with the planet's rotation in mind, jus as orbits from earth take advantage of the direction of the earth's rotation to save energy. We can easily determine the Δv required to convert the hyperbolic capture orbit into an orbit about the capturing planet.

This mission analysis is a bit arbitrary; the spacecraft could also descend to the planet surface or flyby to another destination. But if we desire to calculate the energy requirements for an earth to Mars mission, this is a reasonable formulation and allows a comparison between several options including the more complex trajectories using solutions to Lambert's problem. We can also complete a round-trip mission by remaining in orbit for a specified time and then retracing the process to return to earth. The $\Delta v's$ required to descend to and depart from the planet surface could also be included. These combined Hohmann, escape and capture trajectories are minimum energy trajectories for a given mission and serve as useful benchmarks. When the orbits of the two planets are no longer co-planar, even by a small angle such as the 1.8^o of Mars/earth orbits, the Hohmann transfer is no longer optimum, or even practical, since the plane containing the sun and the two planets at departure and arrival for a Hohmann transfer is substantially tipped with respect to the planetary orbits.

For missions designed to visit several planets, the situation can become very complex as one often tries to take advantage of the gravitational fields of the planets encountered on the way, by entering into their SOI's with the objective of either changing direction or gaining additional impulse. This technique is often referred to as *gravity assist* or flyby.

References

[1] F.J. Hale, *Introduction to Space Flight*, Prentice-Hall, 1994.

[2] M.H. Kaplan, *Modern Spacecraft Dynamics and Control*, John Wiley and Sons, 1976.