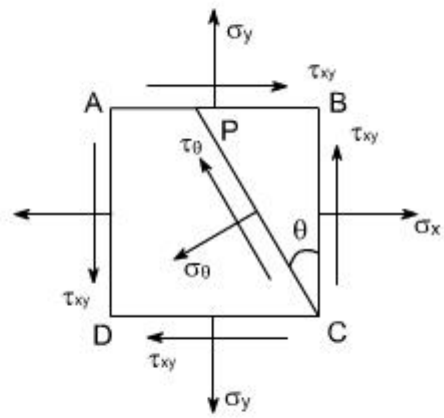
Material subjected to combined direct and shear stresses:

Now consider a complex stress system shown below, acting on an element of material.

The stresses σ_x and σ_y may be compressive or tensile and may be the result of direct forces or as a result of bending.The shear stresses may be as shown or completely reversed and occur as a result of either shear force or torsion as shown in the figure below:



As per the double subscript notation the shear stress on the face BC should be notified as τ_{yx} , however, we have already seen that for a pair of shear stresses there is a set of complementary shear stresses generated such that $\tau_{yx} = \tau_{xy}$

By looking at this state of stress, it may be observed that this state of stress is combination of two different cases:

(i) Material subjected to pure stae of stress shear. In this case the various formulas deserved are as follows

$\sigma_\theta = \tau_{yx} \sin 2\theta$

$\tau_\theta = - \tau_{yx} \cos 2\theta$

(ii) Material subjected to two mutually perpendicular direct stresses. In this case the various formula's derived are as follows.

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta$$

$$\tau_\theta = \frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta$$

To get the required equations for the case under consideration,let us add the respective equations for the above two cases such that

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_\theta = \frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta - \tau_{xy}\cos 2\theta$$

These are the equilibrium equations for stresses at a point. They do not depend on material proportions and are equally valid for elastic and inelastic behaviour

This eqn gives two values of 2θ that differ by 180^0 .Hence the planes on which maximum and minimum normal stresses occurate 90^0 apart.

For σ_θ to be a maximum or minimum $\frac{d\sigma_\theta}{d\theta} = 0$

Now

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\frac{d\sigma_\theta}{d\theta} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta.2 + \tau_{xy}\cos 2\theta.2$$

$$= 0$$

i.e. $-(\sigma_x - \sigma_y)\sin 2\theta + \tau_{xy}\cos 2\theta.2 = 0$

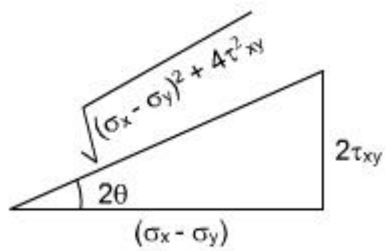
$\tau_{xy}\cos 2\theta.2 = (\sigma_x - \sigma_y)\sin 2\theta$

Thus, $\boxed{\tan 2\theta = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}}$

From the triangle it may be determined

$$\cos 2\theta = \frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$\sin 2\theta = \frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$



Substituting the values of cos2θ and sin2θ in equation (5) we get

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_\theta = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2}.\frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$+ \frac{\tau_{xy}.2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$= \frac{(\sigma_x + \sigma_y)}{2} + \frac{1}{2}.\frac{(\sigma_x - \sigma_y)^2}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$+ \frac{1}{2}\frac{4\tau^2_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

or

$$= \frac{(\sigma_x + \sigma_y)}{2} + \frac{1}{2}.\frac{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}.\frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}.\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$\sigma_\theta = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}.\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$

Hence we get the two values of σ_θ, which are designated σ_1 as σ_2 and respectively,therefore

$$\sigma_1 = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}.\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$

$$\sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}.\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$

The σ_1 and σ_2 are termed as the principle stresses of the system.

Substituting the values of $\cos 2\theta$ and $\sin 2\theta$ in equation (6) we see that

$$\tau_\theta = \frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$= \frac{1}{2}(\sigma_x - \sigma_y)\frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}} - \frac{\tau_{xy}.(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}}$$

$$\tau_\theta = 0$$

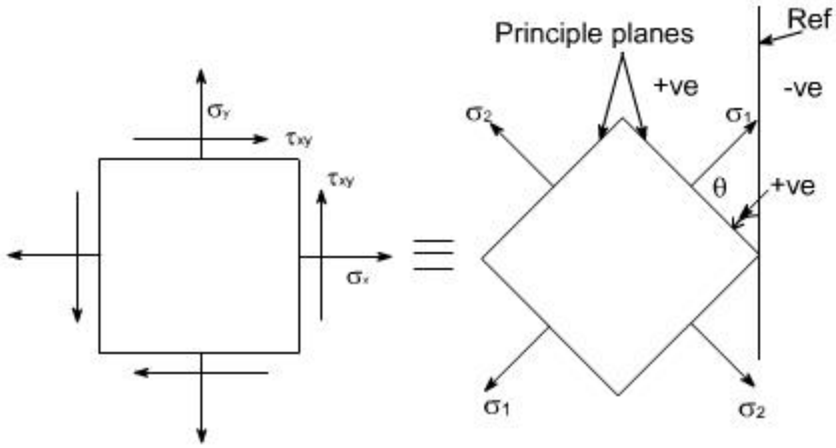
This shows that the values oshear stress is zero on the principal planes.

Hence the maximum and minimum values of normal stresses occur on planes of zero shearing stress. The maximum and minimum normal stresses are called the principal stresses, and the planes on which they act are called principal plane the solution of equation

$$\tan 2\theta_p = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}$$

will yield two values of 2θ separated by 180^0 i.e. two values of θ separated by 90^0 .Thus the two principal stresses occur on mutually perpendicular planes termed principal planes.

Therefore the two – dimensional complex stress system can now be reduced to the equivalent system of principal stresses.



Let us recall that for the case of a material subjected to direct stresses the value of maximum shear stresses

$\tau_{max^m} = \frac{1}{2}(\sigma_x - \sigma_y)$ at $\theta = 45^0$, Thus, for a 2-dimensional state of stress, subjected to principle stresses

$\tau_{max^m} = \frac{1}{2}(\sigma_1 - \sigma_2)$, on substituting the values if σ_1 and σ_2, we get

$$\tau_{max^m} = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2{}_{xy}}$$

Alternatively this expression can also be obtained by differentiating the expression for τ_θ with respect to θ i.e.

$$\tau_\theta = \frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$\frac{d\tau_\theta}{d\theta} = -\frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta.2 + \tau_{xy}\sin 2\theta.2$$

$$= 0$$

or $(\sigma_x - \sigma_y)\cos 2\theta + 2\tau_{xy}\sin 2\theta = 0$

$$\tan 2\theta_s = \frac{(\sigma_y - \sigma_x)}{2\tau_{xy}} = -\frac{(\sigma_x - \sigma_y)}{2\tau_{xy}}$$

$$\tan 2\theta_s = -\frac{(\sigma_x - \sigma_y)}{2\tau_{xy}}$$

Recalling that

$$\tan 2\theta_p = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}$$

Thus,

$$\boxed{\tan 2\theta_p . \tan 2\theta_s = 1}$$

Therefore, it can be concluded that the equation (2) is a negative reciprocal of equation (1) hence the roots for the double angle of equation (2) are 90^0 away from the corresponding angle of equation (1).

This means that the angles that angles that locate the plane of maximum or minimum shearing stresses form angles of 45^0 with the planes of principal stresses.

Futher, by making the triangle we get

$$\cos 2\theta = \frac{2\tau_{xy}}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}}$$

$$\sin 2\theta = \frac{-(\sigma_x - \sigma_y)}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}}$$

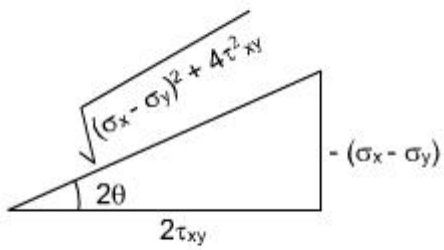
Therefore by substituting the values of $\cos 2\theta$ and $\sin 2\theta$ we have

$$\tau_\theta = \frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$= \frac{1}{2}. - \frac{(\sigma_x - \sigma_y).(\sigma_x - \sigma_y)}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}} - \frac{\tau_{xy}.2\tau_{xy}}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}}$$

$$= -\frac{1}{2}.\frac{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2_{xy}}}$$

$$\tau_\theta = \pm\frac{1}{2}.\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}$$



Because of root the difference in sign convention arises from the point of view of locating the planes on which shear stress act. From physical point of view these sign have no meaning.

The largest stress regard less of sign is always know as maximum shear stress.

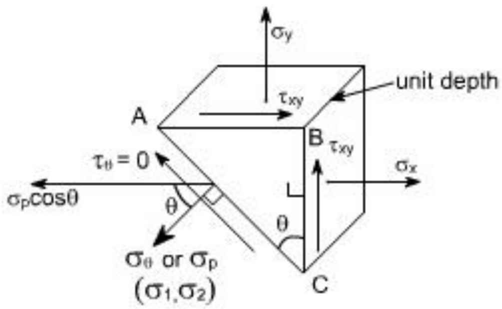
Principal plane inclination in terms of associated principal stress:

We know that the equation $\tan 2\theta_p = \dfrac{2\tau_{xy}}{(\sigma_x - \sigma_y)}$

yields two values of q i.e. the inclination of the two principal planes on which the principal stresses s_1 and s_2 act. It is uncertain,however, which stress acts on which plane unless equation.

$\sigma_\theta = \dfrac{(\sigma_x + \sigma_y)}{2} + \dfrac{(\sigma_x - \sigma_y)}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$ is used and observing which one of the two principal stresses is obtained.

Alternatively we can also find the answer to this problem in the following manner



Consider once again the equilibrium of a triangular block of material of unit depth, Assuming AC to be a principal plane on which principal stresses σ_p acts, and the shear stress is zero.

Resolving the forces horizontally we get:

σ_x .BC . 1 + τ_{xy} .AB . 1 = σ_p . $\cos\theta$. AC   dividing the above equation through by BC we get

$$\sigma_x . + \tau_{xy} \frac{AB}{BC} = \sigma_p . \cos\theta . \frac{AC}{BC}$$

or

$$\sigma_x + \tau_{xy} \tan\theta = \sigma_p$$

Thus

$$\boxed{\tan\theta = \frac{\sigma_p - \sigma_x}{\tau_{xy}}}$$

Source: http://nptel.ac.in/courses/Webcourse-contents/IIT-ROORKEE/strength%20of%20materials/homepage.htm