Introduction to Hydrostatics

Hydrostatics Equation

- The simplified Navier Stokes equation for hydrostatics is a vector equation, which can be split into three components. The convention will be adopted that gravity always acts in the negative z direction. Thus,

  $$\vec{g} = (0, 0, -g)$$

  and the three components of the hydrostatics equation reduce to

  $$\frac{\partial p}{\partial x} = 0$$
  $$\frac{\partial p}{\partial y} = 0$$
  $$\frac{\partial p}{\partial z} = -\rho g$$

  Since pressure is now only a function of z, total derivatives can be used for the z-component instead of partial derivatives. In fact, this equation can be integrated directly from some point 1 to some point 2. Assuming both density and gravity remain nearly constant from 1 to 2 (a reasonable approximation unless there is a huge elevation difference between points 1 and 2), the z-component becomes

  $$\frac{dp}{dz} = -\rho g$$
  $$dp = -\rho g dz$$
  $$\int_{p_1}^{p_2} dp = -\rho g \int_{z_1}^{z_2} dz$$
  $$p_2 - p_1 = -\rho g (z_2 - z_1)$$

- Another form of this equation, which is much easier to remember is

$$p_{below} = p_{above} + \rho g|\Delta z|$$

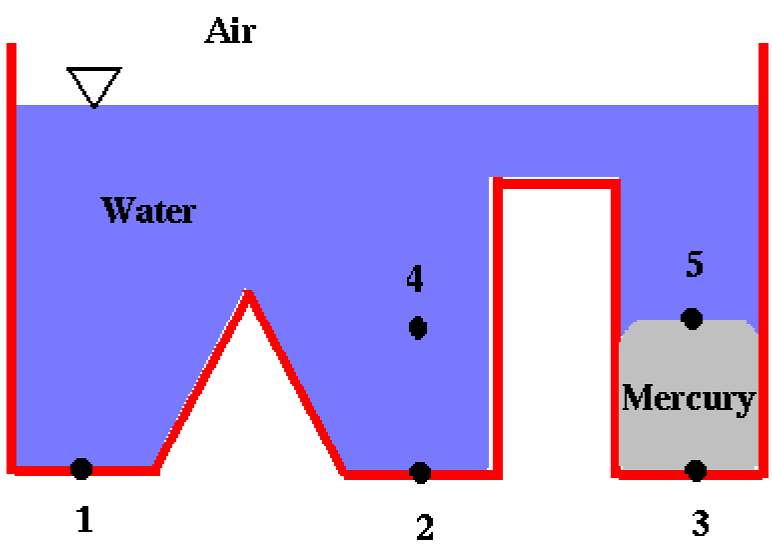
This is the only hydrostatics equation needed. It is easily remembered by thinking about scuba diving. As a diver goes down, the pressure on his ears increases. So, the pressure "below" is greater than the pressure "above."

Some "rules" to remember about hydrostatics

- Recall, for hydrostatics, pressure can be found from the simple equation,

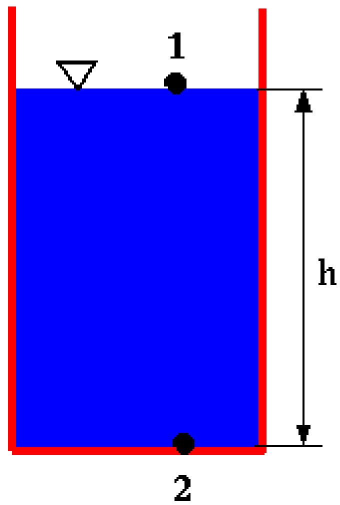
$$p_{below} = p_{above} + \rho g|\Delta z|$$

- There are several "rules" or comments which directly result from the above equation:
- ***If you can draw a continuous line through the same fluid from point 1 to point 2, then $p_1 = p_2$ if $z_1 = z_2$.***
For example, consider the oddly shaped container below:



By this rule, $p_1 = p_2$ and $p_4 = p_5$ since these points are at the same elevation in the same fluid. However, p_2 does not equal p_3 even though they are at the same elevation, because one cannot draw a line connecting these points through the *same* fluid. In fact, p_2 is *less than* p_3 since mercury is denser than water.

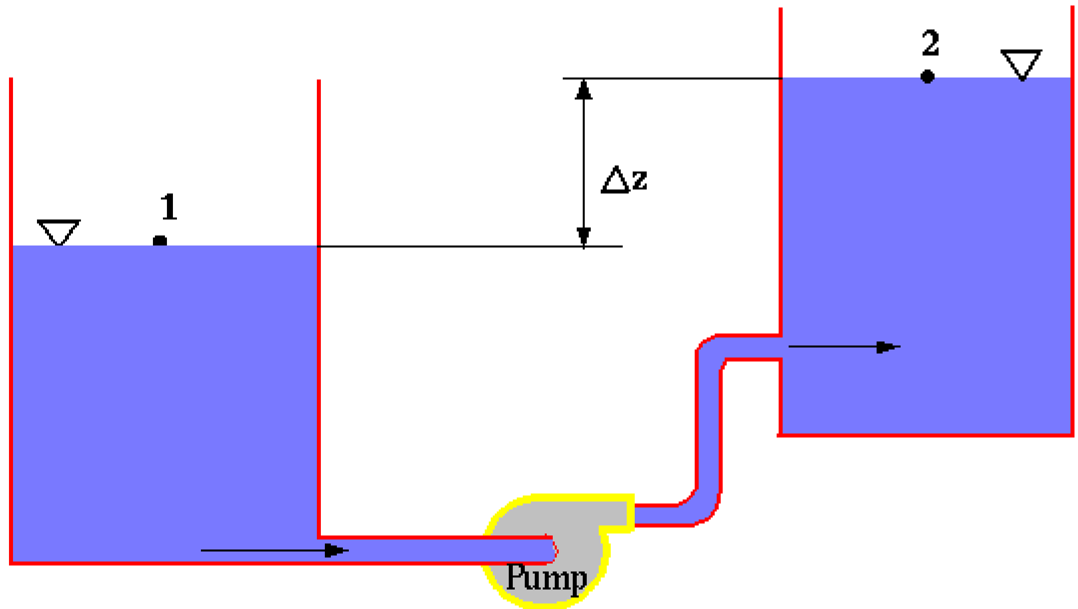
- ***Any free surface open to the atmosphere has atmospheric pressure, p_a.*** (This rule holds not only for hydrostatics, by the way, but for *any* free surface exposed to the atmosphere, whether that surface is moving, stationary, flat, or curved.) Consider the hydrostatics example of a container of water:



The little upside-down triangle indicates a free surface, and means that the pressure there is atmospheric pressure, p_a. In other words, in this example, $p_1 = p_a$. To find the pressure at point 2, our hydrostatics equation is used:

$$p_{below} = p_{above} + \rho g|\Delta z|$$
$$p_2 = p_1 + \rho gh$$
$$p_2 = p_a + \rho gh \quad \text{(absolute pressure)}$$
$$\text{or} \quad p_{2_g} = \rho gh \quad \text{(gage pressure)}$$

- ***In most practical problems, atmospheric pressure is assumed to be constant at all elevations*** (unless the change in elevation is extremely large). Consider the example below, in which water is pumped from one large reservoir to another, as indicated:

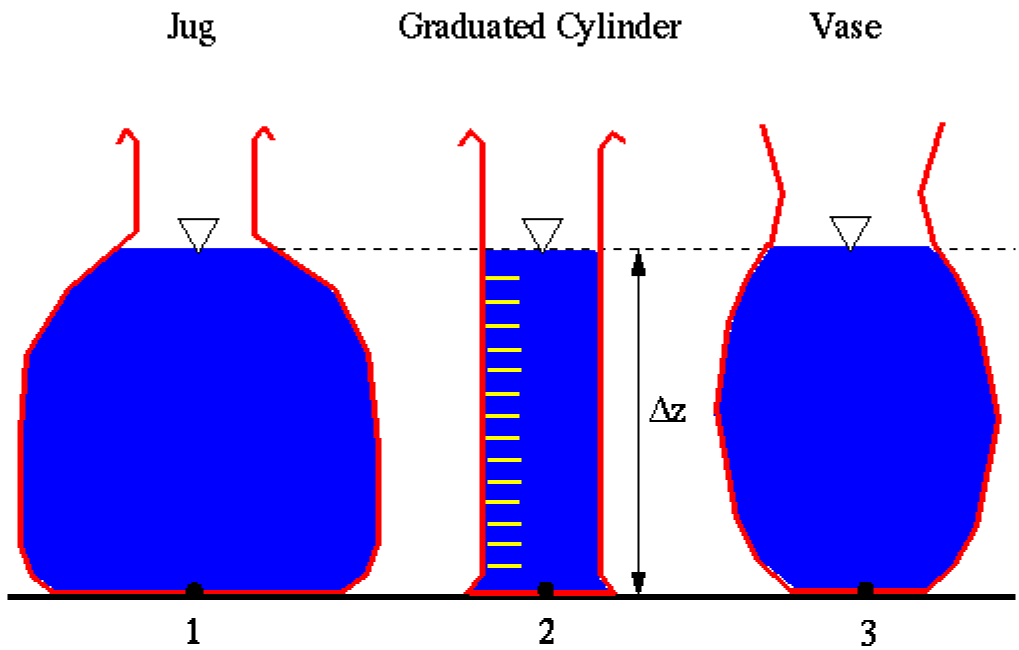


Again, the little upside-down triangle indicates a free surface, and means that the pressure there is atmospheric pressure, p_a. In other words, in this example, $p_1 = p_a$ and $p_2 = p_a$. But since point 2 is higher in elevation than point 1, the local atmospheric pressure at 2 is a little lower than that at point 1. To be precise, our hydrostatics equation must be used to account for the difference in elevation between points 1 and 2:

$$p_{below} = p_{above} + \rho g|\Delta z|$$
$$p_1 = p_2 + \rho_{air} g \Delta z$$

However, since the density of the water in the problem is so much greater than that of the air, it is common to ignore the difference between p_1 and p_2, and call them both the same value of atmospheric pressure, p_a.

- ***The shape of a container does not matter in hydrostatics***. (Except of course for very small diameter tubes, where surface tension becomes important.) Consider the three containers in the figure below:

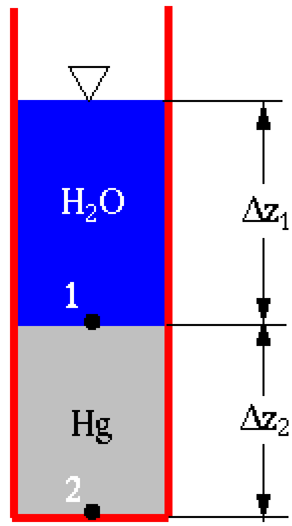


At first glance, it may seem that the pressure at point 3 would be greater than that at point 1 or 2, since the weight of the water is more "concentrated" on the small area at the bottom, but in reality, all three pressures are identical. Use of our hydrostatics equation confirms this conclusion, i.e.

$$p_{below} = p_{above} + \rho g |\Delta z|$$
$$p_1 = p_2 = p_3 = p_a + \rho g \Delta z$$

In all three cases, the thin column of water above the point in question at the bottom is identical. Pressure is a force per unit area, and over a small area at the bottom, that force is due to the weight of the water above it, which is the same in all three cases, regardless of the container shape.

- ***Pressure is constant across a flat fluid-fluid interface***.
  For example, consider the container in the figure below, which is partially filled with mercury, and partially with water:



In this case, our hydrostatics equation must be used twice, once in each of the liquids.

$$
\begin{aligned}
p_{below} &= p_{above} + \rho g|\Delta z| \\
p_1 &= p_a + \rho_{water} g \Delta z_1 \\
p_2 &= p_1 + \rho_{mercury} g \Delta z_2 \\
&= p_a + \rho_{water} g \Delta z_1 + \rho_{mercury} g \Delta z_2
\end{aligned}
$$

Note that if the interface is not flat, but curved, there *will* be a pressure difference across that interface. For example, consider the junction of an air-water interface with a vertical wall. Due to surface tension, the water creeps up the wall, causing the interface to be curved. The pressure at the top of the interface is atmospheric, but the pressure just below the curved portion of the interface is less than atmospheric, due to surface tension in the interface.

Source:

http://www.mne.psu.edu/cimbala/Learning/Fluid/Hydrostatics/hydrostatics.htm