Free and Forced Vibrations of Two Degree of Systems

Introduction:

The simple single degree-of-freedom system can be coupled to another of its kind, producing a mechanical system described by two coupled differential equations; to each mass, there is a corresponding equation of motion. To specify the state of the system at any instant, we need to know time *t* dependence of both coordinates, x_1 and x_2, from which follows the designation two degree-of-freedom system. Further 2 DOF systems find application in vibration absorbers which are very helpful for reducing the vibration levels of the parent system.

Examples: Forging hammer and anvil on ground isolators

IC engine mounted on flexible base (building floor)

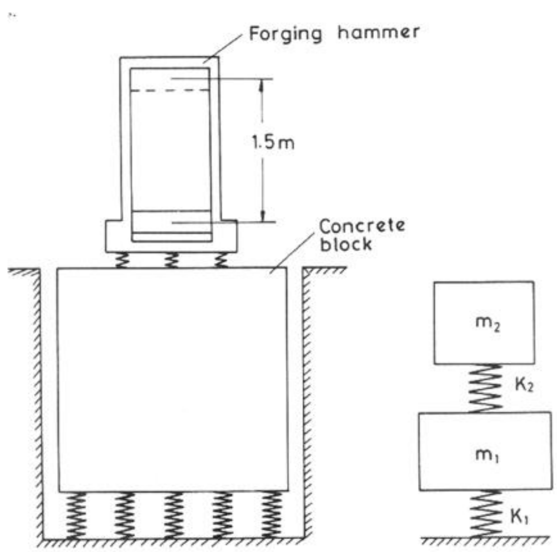


Fig. 1.14 Forging Hammer with anvil [1]

The general equation of motion of for a dynamic for forced vibration is;

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

If we consider $\zeta < 1$ only , the CF is given by :

$$x(t) = e^{-¸ \mu_n t}(A\cos \mu_d t + B \sin \mu_d t)$$

To obtain the PI, we must know the RHS, F(t).

We will consider one type of excitation only :

$$m\ddot{x} + c\dot{x} + kx = F(t) = F_0 \sin \mu t$$

We now need to guess a PI.

When a linear system is subjected to a harmonic excitation of the form *F sinwt*,

- It will respond harmonically at the same frequency.
- There will be a phase lag between the force and the response.

$$Input: \quad F(t) = F_0 \sin \mu t \qquad 0 < \mu < \infty$$

$$Output: \quad x_{PI}(t) = x_0 \sin(\mu t - ¤)$$

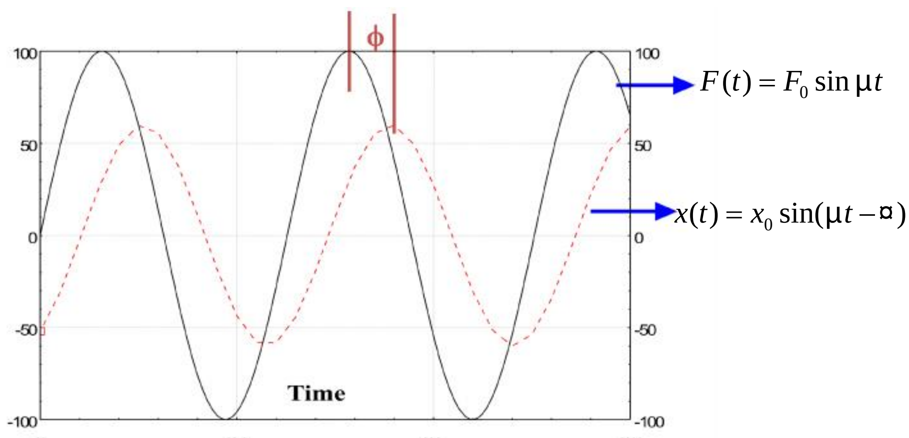


Fig. 1.15 Phase lag

The solution for the steady – state vibration can be found by inserting the PI $x_{PI}(t) = x_0 \sin(\mu t - ¤)$ into the EOM $m\ddot{x} + c\dot{x} + kx = F_0 \sin \mu t$

$$X = \frac{F_0/k}{\sqrt{(1 - m\omega^2/k)^2 + (c\omega/k)^2}}$$

Or,

$$\frac{Xk}{F_0} = \frac{1}{\sqrt{[1-(\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}}$$

And

$$\tan\theta = \frac{(c\omega/k)}{1 - m\,\omega^2/k} = \frac{2\zeta(\omega/\omega_n)}{1-(\omega/\omega_n)^2}$$

And the total solution of equation 1.17 may be obtained for underdamped condition as follows.

$$x(t) = e^{-\zeta\omega_n t}(A\sin\omega_d t + B\cos\omega_d t) + X\sin(\omega t - \theta) \qquad (1.19)$$

here, the values of constants A and B may be determined using the initial condition and the forcing function.

A closer analysis of above equation yields that For a very larger value of t, the transient response (the first term) becomes very small, and hence the term steady state response is assigned to the particular solution (the second term) The value of coefficient of the steady state response, or particular solution becomes large when the excitation frequency is close to the undamped natural frequency,i.e.,$\omega = \omega_n$. This phenomenon is known as resonance and plays a vital role in design,vibration analysis, and testing.

Example 1.4 :

Compute the response of the following system

$$\ddot{x}(t) + 0.4\dot{x}(t) + 4x(t) = \frac{1}{\sqrt{2}}\sin 3t,$$

$x(0) = \frac{-3}{\sqrt{2}}$,

$$\dot{x}(0) = 0$$

First, solve for the particular solution by using the more convenient form of as

$$x_p(t) = X_1\sin 3t + X_2\cos 3t$$

Differentiating x_p yields as follows

$$\dot{x}_p(t) = 3X_1 \cos 3t - 3X_2 \sin 3t$$

$$\ddot{x}_p(\mathrm{t}) = -9\mathrm{X}_1 \sin 3\mathrm{t} - 9\mathrm{X}_2 \cos 3\mathrm{t}$$

Substitution and collection of similar terms yields as

$$\left(-9X_1 - 1.2X_2 + 4X_1 - \frac{1}{\sqrt{2}}\right) \sin 3t + (-9X_2 - 1.2X_1 + 4X_2) \cos 3t = 0$$

Since sine and cosine are independent hence coefficient of sine and cosine should vanish.

$$-9X_1 - 1.2X_2 + 4X_1 - \frac{1}{\sqrt{2}} = 0$$

$$-9X_2 - 1.2X_1 + 4X_2 = 0$$

Solving these equation for X_1 and X_2 and substituting the values, particular solution yields as

$$x_p(t) = -0.134 \sin 3t - 0.032 \cos 3t$$

Given that,

$$\omega_n = 2\,\mathrm{rad/s}, \quad \zeta = \frac{0.4}{2\omega_n} = 0.1 < 1, \quad \omega = \omega_n\sqrt{1 - \zeta^2} = 1.99\ \mathrm{rad/s}$$

Since, the system is underdamped, therefore, the complete solution of the equation yields as

$$x(t) = e^{-\zeta\omega_n t}(A \sin \omega_d t + B \cos \omega_d t) + X_1 \sin \omega t + X_2 \cos \omega t$$

Differentiating the above expression as

$$\dot{x}(t) = e^{-\zeta\omega_n t}(\omega_d A \cos \omega_d t - \omega_d B \sin \omega_d t) + \omega X_1 \cos \omega t - \omega X_2 \sin \omega t - \zeta\omega_n e^{-\zeta\omega_n t}(A \sin \omega_d t + B \cos \omega_d t)$$

Applying the initial condition, the values of the constant A and B may be obtained as

$$x(0) = B + X_2 = \frac{-3}{\sqrt{2}} \Rightarrow B = -X_2 - \frac{3}{\sqrt{2}} = -2.089$$

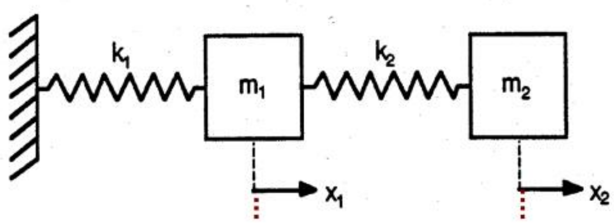
$$\dot{x}(0) = \omega_d A + \omega X_1 - \zeta\omega_n B = 0 \Longrightarrow A = \frac{1}{\omega_d}(\zeta\omega_n B - \omega X_1) = -0.008$$

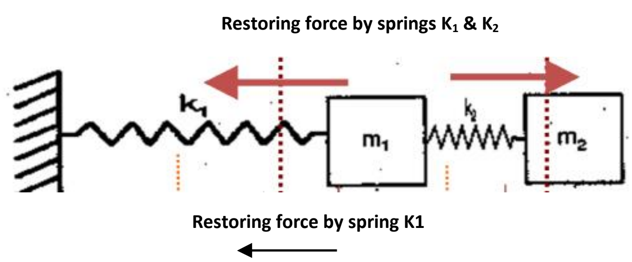
Thus the final desired solution is

$$x(t) = -e^{-0.2t}(0.008\,sin\,1.99t + 2.089\,cos\,1.99t) - 0.134\,sin\,3t - 0.032\,cos\,3t$$

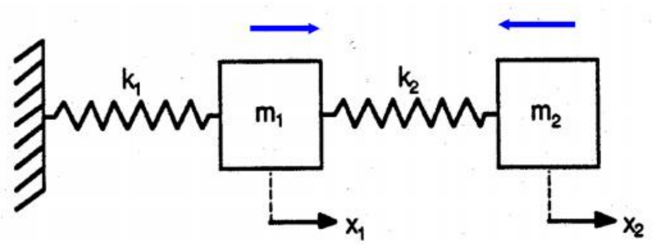
Two-Degree of Freedom system:

There are various steps involved in analyzing the 2-DOF vibrating systems to get the natural frequencies and mode shapes. To resolve the force under spring deflection, the free body diagram is essential required as;





Basic deflection in springs with two spring stiffness (k_1 and k_2)



Displacements of both masses

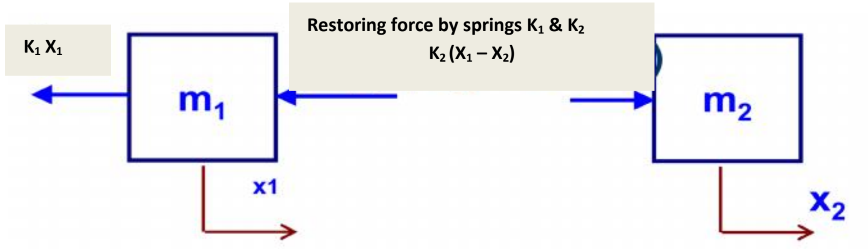


Fig. 1.16 Free Body Diagram for 2 DOF systems

$$\text{Newton's 2nd law}: \quad m\ddot{x} = \sum F$$

Mass 1: Mass 2:

$$\left.\begin{array}{ll} m_1\ddot{x}_1 = -k_1x_1 - k_2(x_1 - x_2) & m_2\ddot{x}_2 = k_2(x_1 - x_2) \\ m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0 & m_2\ddot{x}_2 - k_2x_1 + k_2x_2 = 0 \end{array}\right\} \quad (1.20)$$

$$\text{We have}: \quad \left.\begin{array}{l} m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0 \\ m_2\ddot{x}_2 - k_2x_1 + k_2x_2 = 0 \end{array}\right\} \quad (1.21)$$

$$\text{Remembering that} \quad \ddot{x}_1 = -\omega^2 x_1 \;\; \& \;\; \ddot{x}_2 = -\omega^2 x_2$$

$$-m_1\omega^2 x_1 + (k_1 + k_2)x_1 - k_{2x2} = 0$$

$$-m_2\omega^2 x_2 - k_2x_1 + k_2x_2 = 0$$

$$\left.-\omega^2\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\} \quad (1.22)$$

$$([k] - \omega^2 [M])\{x\} = \{0\} \qquad (1.23)$$

Where the stiffness matrix is

$$[K] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

Mass matrix is $[M] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$

Mode shape $\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$

Here, the Eigen value must be equal to the square of natural frequency. For two degree of freedom system, there must be two natural frequencies and the corresponding two mode shapes exist. The mass & stiffness matrices must be symmetric.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

The main diagonal elements must be positive.

For large n, there are many numerical solution techniques. Use determinant = 0 for small systems as;

$$([K]\text{- } \mu^2[M])\{x\}= \{0\}$$

$$det\ ([K]\ \mu^2\ [M]) = 0 \quad or$$

$$\{x\} = 0$$

For a non-trival solution:

$$det\ ([K]\ \omega^2\ [M]) = 0 \text{ which gives}$$

$$\left.\begin{aligned} &\begin{vmatrix} k_1 + k_2 - \omega^2 & -k_2 \\ -k_2 & k_2 - \omega^2 m_2 \end{vmatrix} = 0 \\ &(k_1 + k_2 - \omega^2)\,(k_2 - \omega^2\, m_2) - k_2^2 = 0 \end{aligned}\right\} \quad (1.24)$$

The above equation gives quadratic in natural frequency, hence two natural frequencies exist, as ω_{n1} and ω_{n2} ($\omega_{n1} \leq \omega_{n2}$)

Insert w_{n1} into $([K]\text{-}w^2[M])\{x\}=\{0\}$

By definition, $det([K]\text{- } w_{n1}{}^2\,[M])=0$

x_1 & x_2 are linearly dependent, but we can obtain x_1/x_2

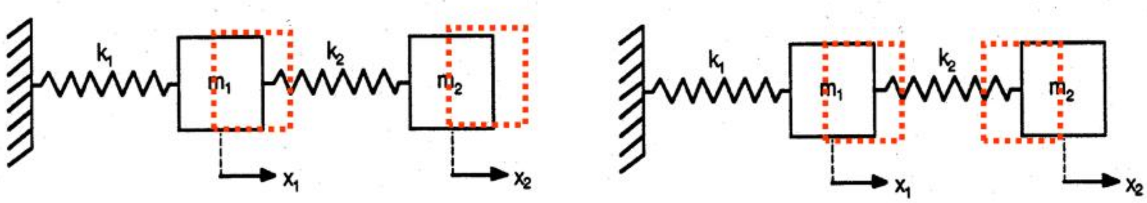
$$\text{Using the previous result}: -m_1\omega^2 x_1 + (k_1 + k_2)x_1 - k_2 x_2 = 0$$

$$\left. Hence: \frac{x_2}{x_1}\right|_{\mu=\mu_{n1}} = \frac{k_1 + k_2 - m_1\mu_{n1}^2}{k_2} \quad (1.25)$$

$$\left. \text{Similarly, for the 2nd mode}: \frac{x_2}{x_1}\right|_{\mu=\mu_{n2}} = \frac{k_1 + k_2 - m_1\mu_{n2}^2}{k_2} \quad (1.26)$$

Assume that, inserting values for m, k, μ gives:

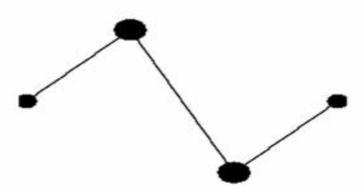
$$\left.\frac{x_2}{x_1}\right|_{\mu=\mu_{n1}} = 1 \quad \& \quad \left.\frac{x_2}{x_1}\right|_{\mu=\mu_{n2}} = -1$$



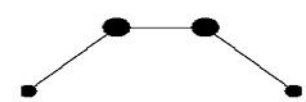
The masses move in phase. X1 and X2 move by +1 unit each.

The masses move out of phase. X1 moves by +1 unit, X2 moves by –1 unit.

Mode Shapes are the Relative Displacements of Bodies at Different Frequencies as shown as;



Mode Shape at First Natural Frequency



Mode Shape at Second Natural Frequency

Two Degree of Freedom Forced Vibrating System:

Two masses are constrained by the springs and two different forces are acting as shown in Fig. 1.17.

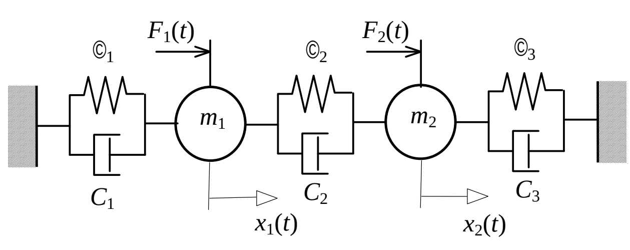


Fig. 1.17 Two Degree of freedom system

Newton's second law for each mass gives

$$m_1 \ddot{x}_1 = F_1\left(x_1(t), x_2(t), \dot{x}_1, \dot{x}_2 \ ,t \right) \qquad \text{(0-27}$$

$$m_2 \ddot{x}_2 = F_2\left(x_1(t), x_2(t), \dot{x}_1, \dot{x}_2 \ ,t \right) \qquad \text{(1.28}$$

$$F_1 = -\copyright_1 x_1(t) - \copyright_2 (x_1(t) - x_2(t)) - C_1 \ \dot{x}_1 - C_2 \ \left(\dot{x}_1 - \dot{x}_2 \right) + F_1(t) \qquad \text{(0-29}$$

$$F_2 = \copyright_2 (x_1(t) - x_2(t)) - \copyright_3 x_2(t) + C_2 \ \left(\dot{x}_1 - \dot{x}_2 \right) - C_3 \ \dot{x}_2 + F_2(t) \qquad \text{(1.30}$$

Equations (1-27) - (1-30) give as;

$$m_1 \ddot{x}_1 + C_1 \dot{x}_1 + C_2\left(\dot{x}_1 - \dot{x}_2 \right) + + \copyright_1 x_1(t) + \copyright_2 \left(x_1(t) - x_2(t) \right) = F_1(t) \qquad \text{(0-31)}$$

$$m_2 \ddot{x}_2 + C_2\left(\dot{x}_1 - \dot{x}_2 \right) + C_3 \dot{x}_2 - \copyright_2 (x_1(t) - x_2(t)) + \copyright_3 x_2(t) = F_2(t) \qquad \text{(1-32)}$$

Matrix and vector notation can be incorporated into (1-31) and (1-32), which is useful for generalizing to an arbitrary number of degrees-of-freedom.

Source:
http://nptel.ac.in/courses/112107088/3