Introduction

Many machines or processes give rise to sound or vibrations which are not periodic, but rather random (stochastic) or transient. Roughness of the contacting surfaces between a wheel and its path, for instance, or between meshing gear teeth, bring about randomly varying vibrations. Turbulence in flowing media gives rise to randomly varying sound. For non-periodic disturbances, a Fourier series decomposition cannot be made; instead, one must use a so-called *Fourier transform.*

The Fourier transform can be developed from the complex Fourier series (9.23, 9.24) of a periodic train of pulses, as shown in Figure 9.11.

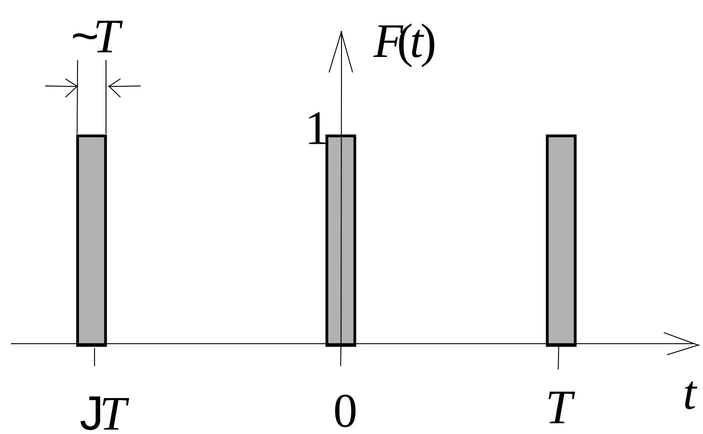


Figure 9.11 Periodic train of pulses with amplitude 1, perod T and pulse width ΔT

Equation (9.23) provides the coefficients of the complex Fourier series,

$$¢_n = \frac{1}{T}\int_{-\Delta T/2}^{\Delta T/2} 1 \cdot e^{-in\omega_0 t}\,dt = \frac{\left(e^{in\omega_0 \Delta T/2} - e^{-in\omega_0 \Delta T/2}\right)}{in\omega_0 T} = \frac{2i\sin(\Delta n\omega_0 T/2)}{in\omega_0 T} \quad (9.26)$$

but $T = 2\ \omega_0$, which implies that

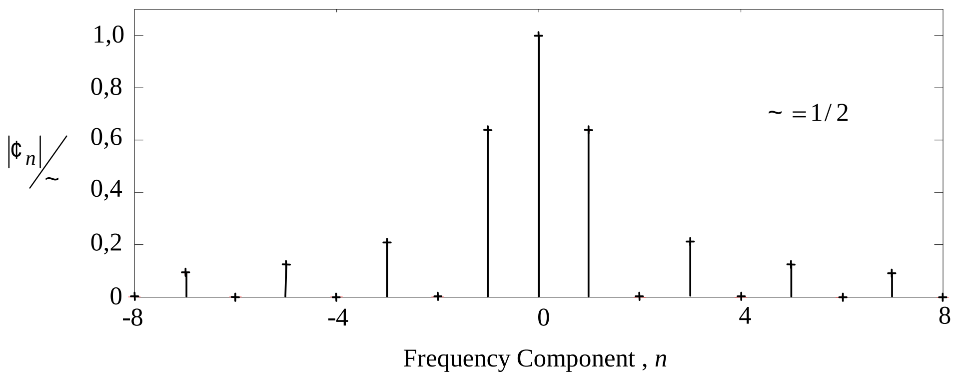
$$¢_n = \Delta\,\frac{\sin(\Delta n ®)}{\Delta n ®} \quad (9.27)$$

The Fourier series of the pulse train becomes

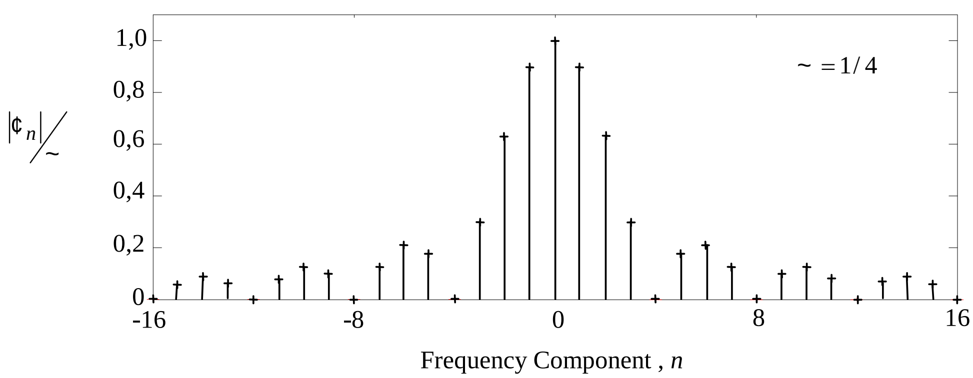
$$\mathbf{F}(t) = \sum_{n=-\infty}^{\infty} \alpha \frac{\sin(\alpha n\pi)}{\alpha n\pi} e^{in\omega_0 t} \tag{9.28}$$

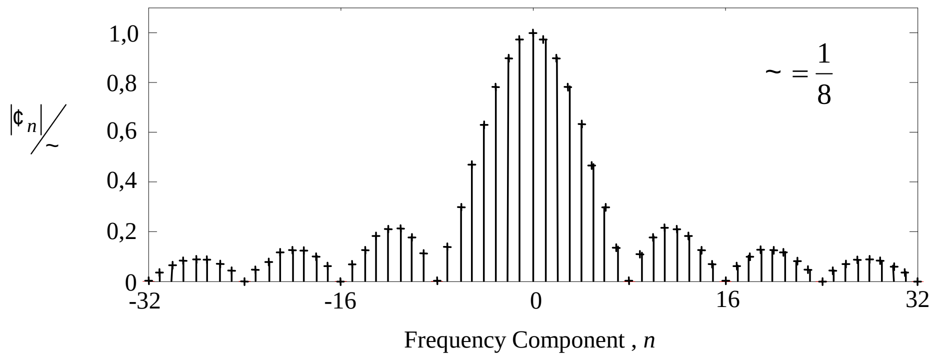
The Fourier series coefficients for the cases $\alpha = 1/2$, $1/4$ and $1/8$ are shown in Figure 9.12. From that figure, it is clear that if the pulses are permitted to glide farther and farther apart in the time domain, $T \to \infty$ and the pulse width αT is held constant, from which it follows that $\alpha T \to 0$, then the spectral lines approach each other in the frequency domain, i.e., become infinitely dense.

a)



b)





c)

Figure 09.12 Fourier series decomposition of a periodic pulse train with constant pulse width aT. For smaller a and increasing period T, the frequency components become all the more densely packed a) α = 1 / 2, b) α = 1 / 4 and c) α = 1 / 8.

To derive a relation for a non-periodic event, we therefore consider the limiting case of the period T becoming infinite. If we substitute in the expression for the Fourier coefficients (9.23) into the Fourier series (9.22), we then obtain

$$F(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{in\mu_0 t}\int_{-T/2}^{T/2} F(t)e^{-in\mu_0 t}\,dt \qquad (9.29)$$

To adapt that to the limiting case when the period T goes to infinity, the interchange $\omega_0 \rightarrow d$~ is made because $\mu_0 = 2\alpha / T$, and $n\mu_0$ transforms into a continuous variable α, i.e., $n\mu_0 \rightarrow 0$ The step size in the summation becomes infinitesimally small, and the summation in equation (9.28) transforms to an integral

$$F(t) = \frac{1}{2\text{®}}\int_{-\infty}^{\infty} e^{in\mu_0 t}\left(\int_{-\infty}^{\infty} F(t)e^{-in\mu_0 t}\,dt\right)d\mu \qquad (9.30)$$

The expression inside the parentheses is identified as the Fourier transform of the signal,

$$\mathbf{F}(\mu) = \int_{-\infty}^{\infty} F(t)e^{-i\mu t}\,dt \qquad (9.31)$$

and the inverse Fourier transform is given by

$$F(t) = \frac{1}{2®}\int_{-\infty}^{\infty} \mathbf{F}(\mu) e^{i\mu t} d\mu \qquad (9.32)$$

The Fourier transform is a complex quantity, which, in the case of $F(t)$ representing a force, has the units N/Hz. In order for $F(t)$ to be real, $\mathbf{F}(-\omega \mathrm{t}) = \mathbf{F^*}(\mu t)$ must hold,

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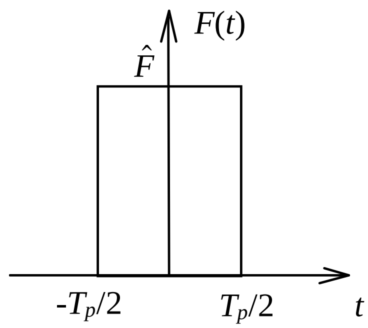
Example 9.2

Calculate the Fourier transform of a single force pulse, with pulse width T_p, as illustrated in the adjacent figure. Applying (9.30) yields



$$\mathbf{F}(\mu) = \int_{-T_p/2}^{T_p/2} \hat{F} e^{-i\mu t} dt =$$

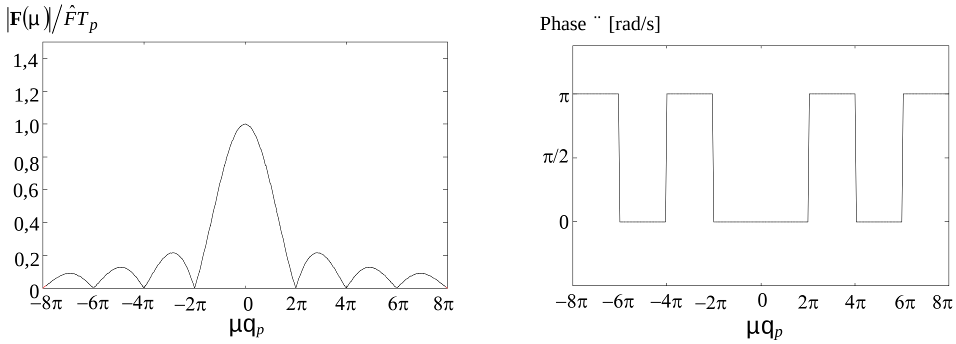
$$= \frac{\hat{F}}{i\mu}\left(e^{i\mu T_p/2} - e^{-i\mu T_p/2}\right) =$$

$$= \hat{F} T_p \frac{\sin\left(\mu T_p/2\right)}{\mu T_p/2}.$$

The amplitude spectrum becomes

$$\left|\mathbf{F}(\mu)\right| = \hat{F} T_p \frac{\left|\sin(\mu T_p / 2)\right|}{\mu T_p / 2}.$$

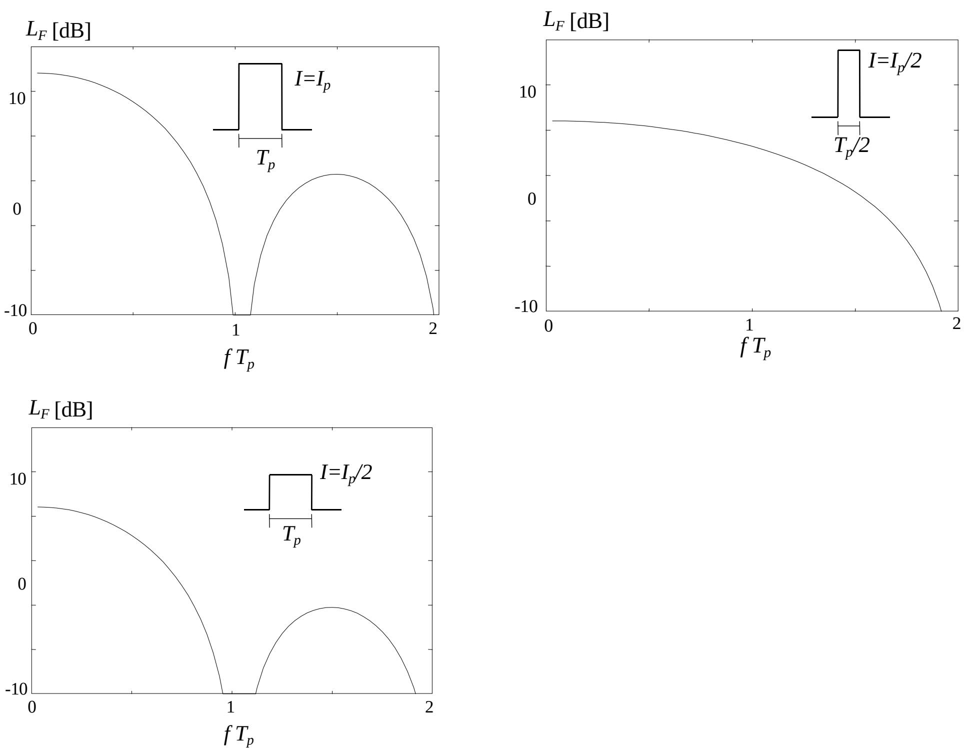
The Fourier transform is real, which implies that the phase spectrum is determined by the sign of sinα T_p. The Fourier transform's *amplitude spectrum* and *phase spectrum* are shown in the figure below, in which a dimensionless frequency ΔT_p has been incorporated. Note that the transform of the rectangular pulse corresponds to the case in which $\Delta T_p \rightarrow 0$. The amplitude spectrum in the figure below is therefore the result obtained as shown in below Fig, in the limit as $\omega\ T_p \rightarrow 0$. The discrete spectrum for the pulse train has, in the case of a single pulse, transformed into a continuous spectrum.



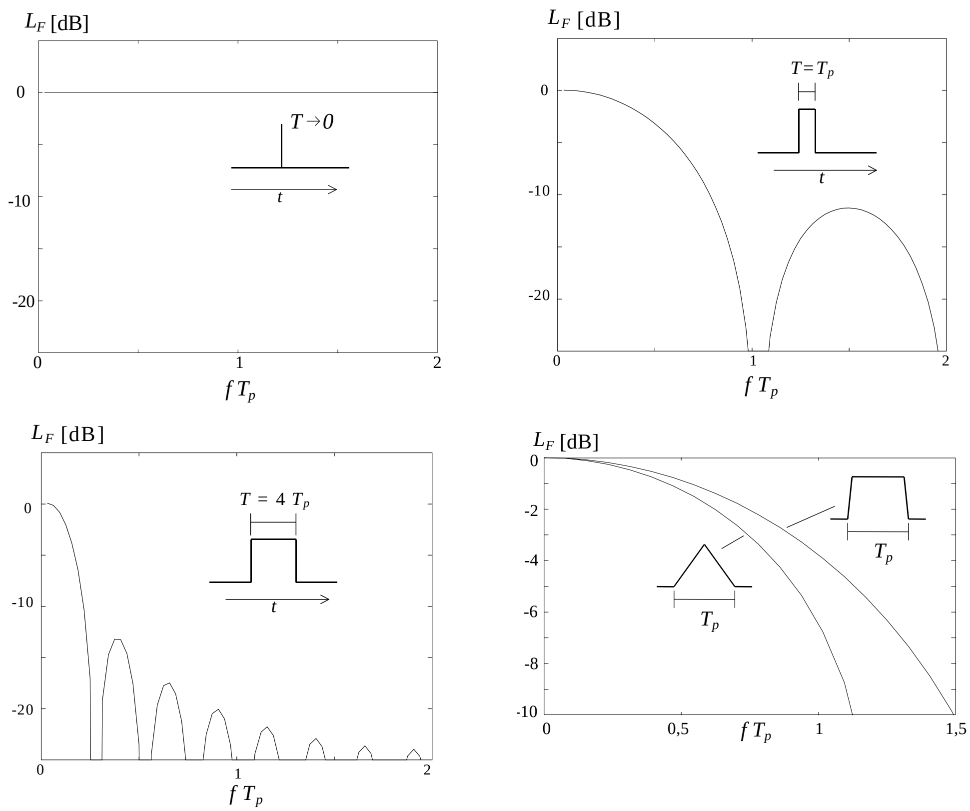
Example 9.3

By analysis of the transient forces in the time and frequency domains, respectively, a number of general conclusions, useful in machine and equipment design, can be drawn.

The smaller the impulse ($I = \int F(t)dt$), the lower the amplitude in the frequency domain. The figure below illustrates the effects of two different modifications to the impulse. Note that a dimensionless frequency $f\,T_p$ is used.



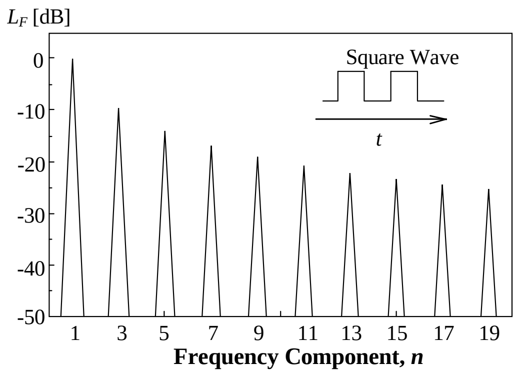
Increased *duration* or *pulse width* T_p in the time domain translates into a lowering of the cutoff frequency (the frequency at which the level has fallen 3 dB with respect to the maximum amplitude). By making the pulse longer (increasing duration), a lower frequency excitation is thereby obtained. That can exploited to shift the excitation into a frequency band which is less disturbing or in which the structure is not as effectively excited. The figure below illustrates that effect.

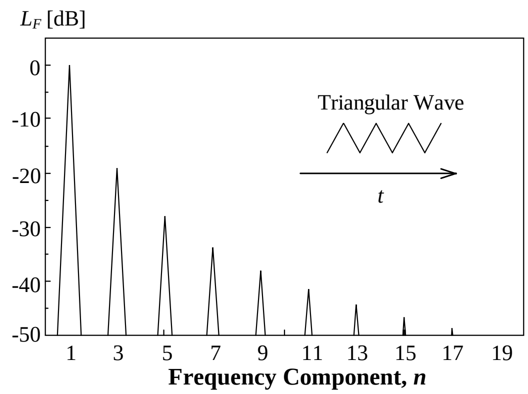


If the rise or fall time of the pulse is lengthened, the amplitude decays more rapidly with frequency above the cutoff frequency; see the adjacent figure. That can be exploited to reduce the high frequency content in the excitation. The same also applies to higher time derivatives. The more rounded and "soft" the excitation is in the time domain, the more rapidly the high frequency content decays.

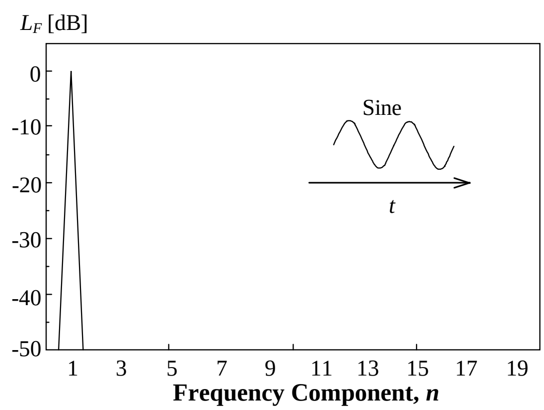
Example 9.4

The nature of the periodic force applications that bring about sound and vibration determines how great the problems that arise are. The figure below shows Fourier series decompositions of a rectangular wave, a triangular wave, and a sine wave.

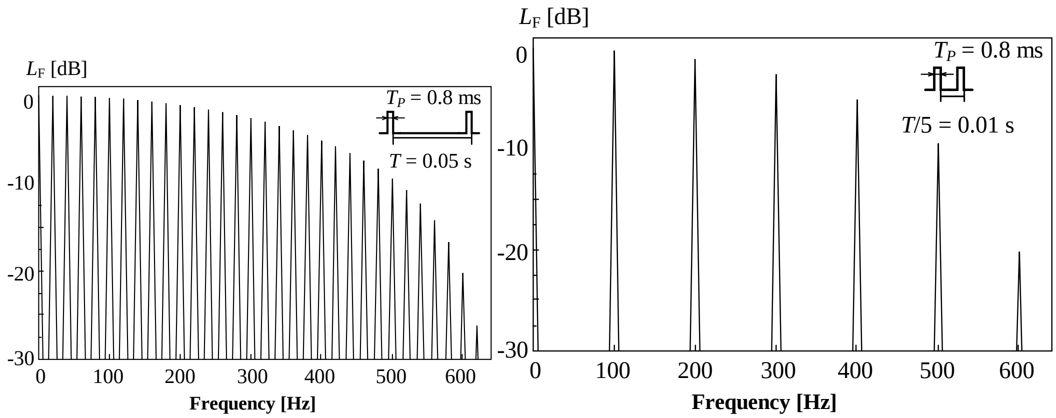




The amplitude of the overtones decays more slowly for the rectangular wave (as $1/n$, where n refers to the n-th frequency component) than for the triangle wave (decays as $1/n^2$), whereas a sine wave only has a single frequency component. Because the overtones often fall in the more disturbing frequency bands, it is a good design principle to always make force applications as soft and "sinusoidal" as possible.



Another phenomenon that occurs in the case of periodic forcing is that the distance f between the frequency components becomes larger, the shorter the period T ; i.e., $f = 1 / T$. That is illustrated in the figure below, for a periodically repeated rectangular pulse.



That fact can be used to minimize the number of frequency components excited in sensitive frequency bands, if it is possible to change the period.

Example 9.5

Measurement of the sound pressure level has been carried out in the third octave bands with center frequencies 800 Hz, 1000 Hz and 1250 Hz, from which the results given in the table below were obtained.

f [Hz]	800	1000	1250
Lp [dB]	73.4	69.8	72.1

We now wish to calculate the sound pressure level for the octave band with the center frequency 1000 Hz.

Solution

Calculate, firstly, the mean squared value of the sound pressure in the third octave bands, using formula,

Then, sum up the mean squared values in accordance with Parseval's relation,

$$\tilde{p}_{oct}^2 = \tilde{p}_1^2 + \tilde{p}_2^2 + \tilde{p}_3^2 \ .$$

f [Hz]	800	1000	1250
$\tilde{p}^2$ [Pa^2]	$8.75\cdot10^{-3}$	$3.82\cdot10^{-3}$	$6.48\cdot10^{-3}$

Calculate the sound pressure level as $L_p = 10\cdot\log(\tilde{p}_{oct}^2 / p_{ref}^2)$.

$$\tilde{p}_{oct}^2 = \tilde{p}_1^2 + \tilde{p}_2^2 + \tilde{p}_3^2 = 8.75\cdot10^{-3} + 3.82\cdot10^{-3} + 6.48\cdot10^{-3} = 1.91\cdot10^{-2},$$

$$L_p = 10\log(\tilde{p}_{oct}^2 / p_{ref}^2) = 10\cdot\log(1.91\cdot10^{-2} / 4\cdot10^{-10}) = 76.8 \text{ dB.}$$

Source:
http://nptel.ac.in/courses/112107088/40