Module
2
Measurement Systems

Lesson 10

Errors and Calibration

Instructional Objectives

At the end of this lesson, the student should be able to:

- Define error
- Classify different types of errors
- Define the terms: mean, variance and standard deviation
- Define the term limiting error for an instrument
- Estimate the least square straight line from a set of dispersed data
- Distinguish between the terms: single point calibration and two point calibration.

Introduction

Through measurement, we try to obtain the value of an unknown parameter. However this measured value cannot be the actual or true value. If the measured value is very close to the true value, we call it to be a very accurate measuring system. But before using the measured data for further use, one must have some idea how accurate is the measured data. So error analysis is an integral part of measurement. We should also have clear idea what are the sources of error, how they can be reduced by properly designing the measurement methodology and also by repetitive measurements. These issues have been dwelt upon in this lesson. Besides, for maintaining the accuracy the readings of the measuring instrument are frequently to be compared and adjusted with the reading of another standard instrument. This process is known as calibration. We will also discuss about calibration in details.

Error Analysis

The term *error* in a measurement is defined as:

$$Error = Instrument\ reading - true\ reading. \qquad (1)$$

Error is often expressed in percentage as:

$$\%\,Error = \frac{Instrument\ reading - true\ reading}{true\ reading} X100 \qquad (2)$$

The errors in instrument readings may be classified in to three categories as:

1. *Gross errors*
2. *Systematic errors*
3. *Random Errors.*

Gross errors arise due to human mistakes, such as, reading of the instrument value before it reaches steady state, mistake of recording the measured data in calculating a derived measured, etc. Parallax error in reading on an analog scale is also is also a source of gross error. Careful reading and recording of the data can reduce the gross errors to a great extent.

Systematic errors are those that affect all the readings in a particular fashion. Zero error, and bias of an instrument are examples of systematic errors. On the other hand, there are few errors, the cause of which is not clearly known, and they affect the readings in a random way. This type of errors is known as *Random error*. There is an important difference between the systematic errors and random errors. In most of the case, the systematic errors can be corrected by calibration, whereas the random errors can never be corrected, the can only be reduced by averaging, or error limits can be estimated.

Systematic Errors

Systematic errors may arise due to different reasons. It may be due to the shortcomings of the instrument or the sensor. An instrument may have a zero error, or its output may be varying in a nonlinear fashion with the input, thus deviating from the ideal linear input/output relationship. The amplifier inside the instrument may have input offset voltage and current which will contribute to zero error. Different nonlinearities in the amplifier circuit will also cause error due to nonlinearity. Besides, the systematic error can also be due to improper design of the measuring scheme. It may arise due to the loading effect, improper selection of the sensor or the filter cut off frequency. Systematic errors can be due to environmental effect also. The sensor characteristics may change with temperature or other environmental conditions.

The major feature of systematic errors is that the sources of errors are recognisable and can be reduced to a great extent by carefully designing the measuring system and selecting its components. By placing the instrument in a controlled environment may also help in reduction of systematic errors. They can be further reduced by proper and regular calibration of the instrument.

Random Errors

It has been already mentioned that the causes of random errors are not exactly known, so they cannot be eliminated. They can only be reduced and the error ranges can be estimated by using some statistical operations. If we measure the same input variable a number of times, keeping all other factors affecting the measurement same, the same measured value would not be repeated, the consecutive reading would rather differ in a random way. But fortunately, the deviations of the readings normally follow a particular distribution (mostly normal distribution) and we may be able to reduce the error by taking a number of readings and averaging them out.

Few terms are often used to chararacterize the distribution of the measurement, namely,

$$Mean\,Value\;\; \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad (3)$$

where n is the total number of readings and x_i is the value of the individual readings. It can be shown that the mean value is the most probable value of a set of readings, and that is why it has a very important role in statistical error analysis. The *deviation* of the individual readings from the mean value can be obtained as :

$$Deviation \qquad d_i = x_i - \bar{x} \qquad (4)$$

We now want to have an idea about the deviation, i.e., whether the individual readings are far away from the mean value or not. Unfortunately, the *mean of deviation* will not serve the purpose, since,

$$Mean\,of\;\,deviation = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x}) = \bar{x} - \frac{1}{n}(n\bar{x}) = 0$$

So instead, *variance* or the *mean square deviation* is used as a measure of the deviation of the set of readings. It is defined as:

$$Variance \quad V = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \sigma^2 \tag{5}$$

The term σ is denoted as *standard deviation.* It is to be noted that in the above expression, the averaging is done over *n-1* readings, instead of *n* readings. The above definition can be justified, if one considers the fact that if it is averaged over *n*, the variance would become zero when *n*=1 and this may lead to some misinterpretation of the observed readings. On the other hand the above definition is more consistent, since the variance is undefined if the number of reading is *one*. However, for a large number of readings (*n>30*), one can safely approximate the variance as,

$$Variance\, V = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \sigma^2 \tag{6}$$

The term *standard deviation* is often used as a measure of uncertainty in a set of measurements. Standard deviation is also used as a measure of quality of an instrument. It has been discussed in Lesson-3 that *precision*, a measure of reproducibility is expressed in terms of standard deviation.

Propagation of Error

Quite often, a variable is estimated from the measurement of two parameters. A typical example may be the estimation of power of a d.c circuit from the measurement of voltage and current in the circuit. The question is that how to estimate the uncertainty in the estimated variable, if the uncertainties in the measured parameters are known. The problem can be stated mathematically as,

Let $\quad y = f(x_1, x_2, ..., x_n) \qquad (7)$

If the uncertainty (or deviation) in x_i is known and is equal to Δx_i , $(i = 1,2,..n)$, what is the overall uncertainty in the term *y*?

Differentiating the above expression, and applying Taylor series expansion, we obtain,

$$\Delta y = \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2 + ... + \frac{\partial f}{\partial x_n}\Delta x_n \tag{8}$$

Since Δx_i can be either +ve or –ve in sign, the maximum possible error is when all the errors are positive and occurring simultaneously. The term *absolute error* is defined as,

$$Absolute\,error: |\Delta y| = \frac{\partial f}{\partial x_1}|\Delta x_1| + \frac{\partial f}{\partial x_2}|\Delta x_2| + ... + \frac{\partial f}{\partial x_n}|\Delta x_n| \tag{9}$$

But this is a very unlikely phenomenon. In practice, $x_1, x_2, ..., x_n$ are independent and all errors do not occur simultaneously. As a result, the above error estimation is very conservative. To alleviate this problem, the cumulative error in *y* is defined in terms of the standard deviation. Squaring equation (8), we obtain,

$$(\Delta y)^2 = \left(\frac{\partial f}{\partial x_1}\right)^2 (\Delta x_1)^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 (\Delta x_2)^2 + ... + 2\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2}.(\Delta x_1 \Delta x_2) + ... \tag{10}$$

If the variations of $x_1, x_2, ...$ are independent, positive value of one increment is equally likely to be associated with the negative value of another increment, so that the some of all the cross product terms can be taken as zero, in repeated observations. We have already defined *variance* V as the mean squared error. So, the mean of $(\Delta y)^2$ for a set of repeated observations, becomes the variance of y, or

$$V(y) = \left(\frac{\partial f}{\partial x_1}\right)^2 V(x_1) + \left(\frac{\partial f}{\partial x_2}\right)^2 V(x_2) + \ldots\ldots \qquad (11)$$

So the standard deviation of the variable y can be expressed as:

$$\sigma(y) = \left[\left(\frac{\partial f}{\partial x_1}\right)^2 \sigma^2(x_1) + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma^2(x_2) + \ldots\ldots\right]^{\frac{1}{2}} \qquad (12)$$

Limiting Error

Limiting error is an important parameter used for specifying the accuracy of an instrument. The limiting error (or guarantee error) is specified by the manufacturer to define the maximum limit of the error that may occur in the instrument. Suppose the accuracy of a 0-100V voltmeter is specified as 2% of the full scale range. This implies that the error is guaranteed to be within $\pm$2V for any reading. If the voltmeter reads 50V, then also the error is also within $\pm$2V. As a result, the accuracy for this reading will be $\frac{2}{5}\times 100 = 4\%$. If the overall performance of a measuring system is dependent on the accuracy of several independent parameters, then the limiting or guarantee error is decided by the absolute error as given in the expression in (9). For example, if we are measuring the value of an unknown resistance element using a wheatstone bridge whose known resistors have specified accuracies of 1%, 2% and 3% respectively, then,

Since $R_x = \frac{R_1 R_2}{R_3}$, we have,

$$\Delta R_x = \frac{R_2}{R_3}\Delta R_1 + \frac{R_1}{R_3}\Delta R_2 - \frac{R_1 R_2}{R_3^2}\Delta R_3$$

or, $$\frac{\Delta R_x}{R_x} = \frac{\Delta R_1}{R_1} + \frac{\Delta R_2}{R_2} - \frac{\Delta R_3}{R_3}$$

Then following the logic given to establish (9), the absolute error is computed by taking the positive values only and the errors will add up; as a result the limiting error for the unknown resistor will be 6%.

Importance of the Arithmetic Mean

It has been a common practice to take a number of measurements and take the arithmetic mean to estimate the average value. But the question may be raised: *why mean?* The answer is: *The most probable value of a set of dispersed data is the arithmetic mean.* The statement can be substantiated from the following proof.

Let $x_1, x_2, x_3, \ldots., x_n$ be a set of n observed data. Let X be the central value (not yet specified).

So the deviations from the central value are $(x_1 - X), (x_2 - X), \ldots.(x_n - X)$.

The sum of the square of the deviations is:

$$S_{sq} = (x_1 - X)^2 + (x_2 - X)^2 + ... + (x_n - X)^2$$

$$= x_1^2 + x_2^2 + ... + x_n^2 - 2X(x_1 + x_2 + ... + x_n) + nX^2$$

So the problem is to find X so that S_{sq} is minimum. So,

$$\frac{dS_{sq}}{dX} = -2(x_1 + x_2 + ... + x_n) + 2nX = 0$$

or,

$$X = \frac{1}{n}(x_1 + x_2 + ... + x_n) = \bar{x}$$

So the arithmetic mean is the central value in the least square sense. If we take another set of readings, we shall reach at a different mean value. But if we take a large number of readings, definitely we shall come very close to the actual value (or universal mean). So the question is, how to determine the deviations of the different set of mean values obtained from the actual value?

Standard deviation of the mean

Here we shall try to find out the standard deviation of the mean value obtained from the universal mean or actual value.

Consider a set of n number of readings, $x_1, x_2, x_{3,}...., x_n$. The mean value of this set expressed as:

$$\bar{x} = \frac{1}{n}(x_1 + x_2 + ... + x_n) = f(x_1 + x_2 + ... + x_n)$$

Using (11) for the above expression, we can write:

$$V(\bar{x}) = \left(\frac{\partial f}{\partial x_1}\right)^2 V(x_1) + \left(\frac{\partial f}{\partial x_2}\right)^2 V(x_2) + ... + \left(\frac{\partial f}{\partial x_n}\right)^2 V(x_n)$$

$$= \frac{1}{n^2}\left[V(x_1) + V(x_2)..... + V(x_n)\right]$$

Now the standard deviation for the readings $x_1, x_2, ..., x_n$ is defined as:

$$\sigma = \left[\frac{1}{n}\left[V(x_1) + V(x_2)..... + V(x_n)\right]\right]^{1/2}$$, where n is large.

Therefore,

$$V(\bar{x}) = \frac{1}{n^2}(n.\sigma^2) = \frac{\sigma^2}{n}$$

Hence, the standard deviation of the mean,

$$\sigma(\bar{x}) = \frac{\sigma}{\sqrt{n}} \qquad (13)$$

which indicates that the precision can be increased, (i.e. $\sigma(\bar{x})$ reduced) by taking more number of observations. But the improvement is slow due to the $\sqrt{n}$ factor.

Example: Suppose, a measuring instrument produces a random error whose standard deviation is 1%. How many measurements should be taken and averaged, in order to reduce the standard deviation of the mean to <0.1%?

Solution: In this case,

$$\frac{\sigma}{\sqrt{n}} < 0.1; \quad \therefore \sqrt{n} > \frac{1}{0.1} = 10; \quad or, \; n > 100.$$

Least square Curve Fitting

Often while performing experiments, we obtain a set of data relating the input and output variables (e.g. resistance vs. temperature characteristics of a resistive element) and we want to fit a smooth curve joining different experimental points. Mathematically, we want to fit a polynomial over the experimental data, such that the sum of the square of the deviations between the experimental points and the corresponding points of the polynomial is minimum. The technique is known as least square curve fitting. We shall explain the method for a straight line curve fitting. A typical case of least square straight line fitting for a set of dispersed data is shown in Fig. 1. We want to obtain the best fit straight line out of the dispersed data shown.

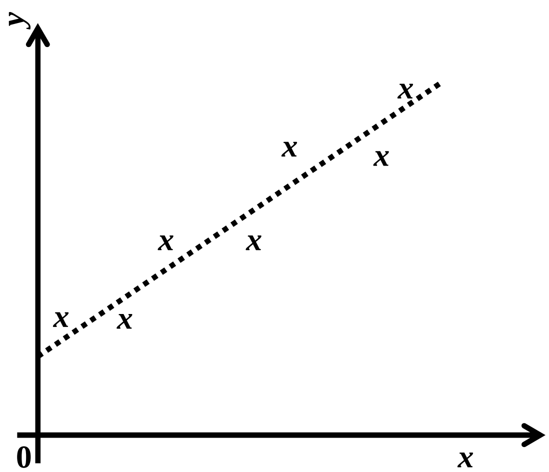

Fig. 1 Least square straight line fitting.

Suppose, we have a set of n observed data $(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)$. We want to estimate a straight line

$$y^* = a_0 + a_1 x \tag{14}$$

such that the integral square error is minimum. The unknowns in the estimated straight line are the constants a_0 and a_1. Now the error in the estimation corresponding to the *i*-th reading:

$$e_i = y_i - y^* = y_i - a_0 - a_1 x_i$$

The integral square error is given by,

$$S_e = \sum_{i=1}^{n} e_i^2 \quad = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i)^2$$

For minimum integral square error,

$$\frac{\partial S_e}{\partial a_0} = 0\,, \quad \frac{\partial S_e}{\partial a_1} = 0$$

or,

$$\frac{\partial S_e}{\partial a_0} = -2\sum_{i=1}^{n}(y_i - a_0 - a_1 x_i) \quad = 0 \tag{15}$$

and
$$\frac{\partial S_e}{\partial a_1} = -2\sum_{i=1}^{n} x_i(y_i - a_0 - a_1 x_i) \quad = 0 \tag{16}$$

From (15) and (16), we obtain,

$$\sum_{i=1}^{n} y_i - a_0.n - a_1 \sum_{i=1}^{n} x_i = 0\,,$$

$$\sum_{i=1}^{n} x_i y_i - a_0 \sum_{i=1}^{n} x_i - a_1 \sum_{i=1}^{n} x_i^2 = 0\,.$$

Solving, we obtain,

$$a_1 = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i{}^2 - (\sum_{i=1}^{n} x_i)^2}$$

or,

$$a_1 = \frac{\frac{1}{n}\sum_{i=1}^{n} x_i y_i - \bar{x}.\bar{y}}{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2} \tag{17}$$

where $\bar{x}$ and $\bar{y}$ are the mean values of the experimental readings x_i and y_i respectively. Using (14), we can have,

$$a_0 = \bar{y} - a_1 \bar{x} \tag{18}$$

Calibration and error reduction

It has already been mentioned that the random errors cannot be eliminated. But by taking a number of readings under the same condition and taking the mean, we can considerably reduce the random errors. In fact, if the number of readings is very large, we can say that the mean value will approach the true value, and thus the error can be made almost zero. For finite number of readings, by using the statistical method of analysis, we can also estimate the range of the measurement error.

On the other hand, the systematic errors are well defined, the source of error can be identified easily and once identified, it is possible to eliminate the systematic error. But even for a simple instrument, the systematic errors arise due to a number of causes and it is a tedious process to identify and eliminate all the sources of errors. An attractive alternative is to calibrate the instrument for different known inputs.

Calibration is a process where a known input signal or a series of input signals are applied to the measuring system. By comparing the actual input value with the output indication of the system, the overall effect of the systematic errors can be observed. The errors at those calibrating points are then made zero by *trimming* few adjustable components, by using calibration charts or by using software corrections.

Strictly speaking, calibration involves comparing the measured value with the *standard instruments* derived from comparison with the primary standards kept at Standard Laboratories. In an actual calibrating system for a pressure sensor (say), we not only require a standard pressure measuring device, but also a *test-bench*, where the desired pressure can be generated at different values. The calibration process of an acceleration measuring device is more difficult, since, the desired acceleration should be generated on a body, the measuring device has to be mounted on it and the actual value of the generated acceleration is measured in some indirect way.

The calibration can be done for all the points, and then for actual measurement, the true value can be obtained from a *look-up table* prepared and stored before hand. This type of calibration, is often referred as *software calibration*. Alternatively, a more popular way is to calibrate the instrument at one, two or three points of measurement and trim the instrument through independent adjustments, so that, the error at those points would be zero. It is then expected that error for the whole range of measurement would remain within a small range. These types of calibration are known as single-point, two-point and three-point calibration. Typical input-output characteristics of a measuring device under these three calibrations are shown in fig.2.

The single-point calibration is often referred as offset adjustment, where the output of the system is forced to be zero under zero input condition. For electronic instruments, often it is done automatically and is the process is known as *auto-zero* calibration. For most of the field instruments calibration is done at two points, one at zero input and the other at full scale input. Two independent adjustments, normally provided, are known as *zero* and *span* adjustments.

One important point needs to be mentioned at this juncture. The characteristics of an instrument change with time. So even it is calibrated once, the output may deviate from the calibrated points with time, temperature and other environmental conditions. So the calibration process has to be repeated at regular intervals if one wants that it should give accurate value of the measurand through out.

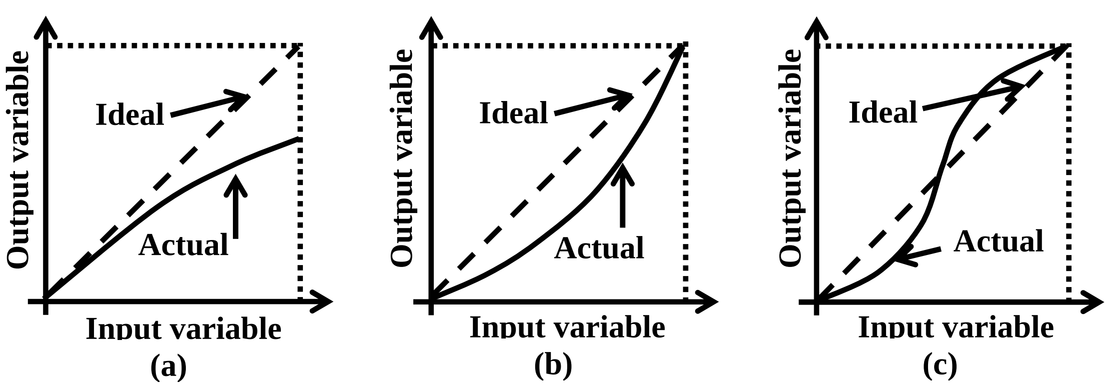

Fig. 2 (a) single point calibration, (b) two point calibration, (c) three point calibration

Conclusion

Errors and calibration are two major issues in measurement. In fact, knowledge on measurement remains incomplete without any comprehensive idea on these two issues. In this chapter we have tried to give a brief overview about errors and calibration. The terms error and limiting error have been defined and explained. The different types of error are also classified. The methods for reducing random errors through repetitive measurements are explained. We have also discussed the least square straight line fitting technique. The propagation of error is also discussed. However, though the importance of mean and standard deviation has been elaborated, for the sake of brevity, the normal distribution, that random errors normally follows, has been left out.
The performance of an instrument changes with time and many other physical parameters. In order to ensure that the instrument reading will follow the actual value within reason accuracy, calibration is required at frequent intervals. In this process we compare and adjust the instrument readings to give true values at few selected readings. Different methods of calibration, e.g., single point calibration, two point calibration and three point calibration have been explained.

References

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Review Questions

1. Define error. A temperature indicator reads 189.8^0C when the actual temperature is 195.5^0C. Find the percentage error in the reading.
2. Distinguish between gross error and systematic error. Write down two possible sources of systematic error.
3. Explain the term limiting error. In a multiple range instrument it is always advisable to take a reading where the indication is near the full scale: justify.
4. The most probable value of a set of dispersed data is the arithmetic mean: justify.
5. The resistance value at a temperature t of a metal wire, R_t is given by the expression, $R_t = R_0(1+\alpha t)$ where, R_0 is the resistance at 0^oC, and α is the resistance temperature coefficient. The resistance values of the metal wire at different temperatures have been tabulated as given below. Obtain the values of R_0 and α using least square straight line fitting.

Temperature (^{o}C)	20	40	60	80	100
Resistance (ohm)	107.5	117.0	117.0	128.0	142.5

6. Most of the instruments have zero and span adjustments. What type of calibration is it?
7. Explain three point calibration and its advantage over the other types of calibration.

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