Elasticity

When external force is applied on a crystalline body, the atoms within get displaced. The atoms get displaced to new positions according to the external force. Atoms respond to the force by changing their displacements according to external force. The extent of atomic displacement depends on the inherent property of the interatomic binding energy. Atoms will occupy new positions such that the external force and internal interatomic force balance each other. This is for small displacements. We can say that the force on a bond is proportional to inter atomic displacement.

1.1 Normal and shear elastic deformations

In crystalline solids the extent to which elastic strain happens is very small, of the order of 0.5% or less. Therefore, we can assume that engineering strain and true strain are identical. Also, we can assume that strain is proportional to stress. This is valid for isotropic materials and for small strains. When a solid gets elastically deformed, it can undergo change in its length, it can undergo shape change or it can undergo rotation.

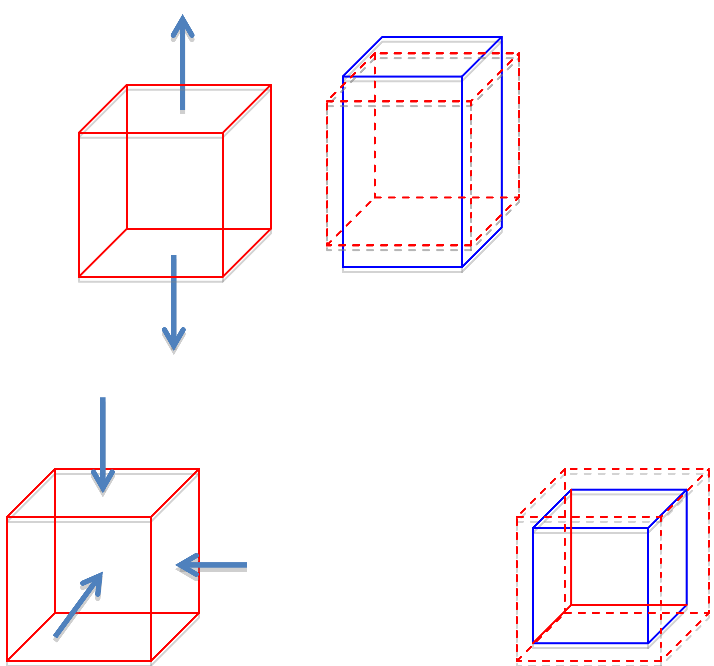


Fig. 3.1.1: Homogeneous Elastic deformations

Shear deformation is caused by shear stress. Shear deformation is dependent on the nature of stress applied. In shear, a material may undergo rotation as well as distortion.

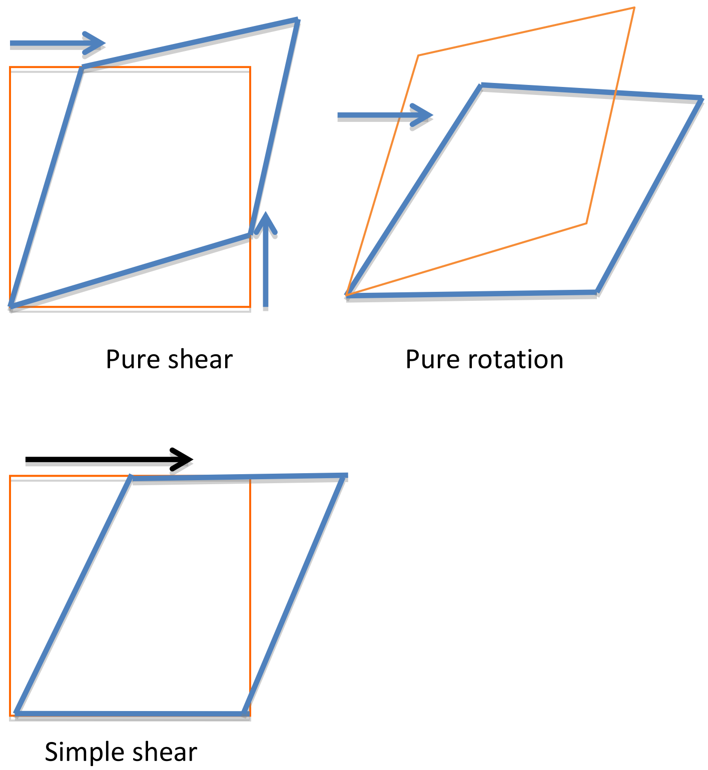


Fig. 3.1.2: Types of shear deformations

From the above figures we can see that the deformation in simple shear can be equivalent to combined pure shear deformation and rotation.

Shear strain can be produced only by shear stress. Therefore for elastic shear we can write Hooke's law as:

$$\gamma = \tau / G$$

G is shear modulus, which is a function of Elastic modulus.

G = $\frac{E}{2(1+\nu)}$

The bulk modulus B is given from the expression for volumetric strain:

$$\frac{\Delta V}{V} = \frac{\sigma_m}{B}$$

B is also dependent on E

B = $\frac{E}{3(1-2\nu)}$

We understand from this equation the following:

When Poisson's ratio =0, the Bulk modulus B = E/3

If Poisson ratio = ½ then B tends to infinity. This is the case of rigid plastic materials, which are incompressible.

If Poisson ratio is > ½ we get B as negative, which is not possible.

For small strains, we can write:

$$\frac{\Delta V}{V} = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

For tensile deformation, we write for uniaxial state of stress:

$$\varepsilon_x = \frac{\sigma_x}{E}$$

E is Young's modulus or elastic modulus.

Young's modulus is a material property, which depends on the nature of bond and binding energy – energy trough. High melting point materials have high Young's modulus values and vice-versa.

Generally, Young's modulus decreases with increase in temperature from room temperature upto melting temperature.

1.2 Thermal strain:

Suppose a material undergoes change in its temperature, ΔT. Such change in temperature may induce strain in the material. The thermal strain is given by:

e = $\frac{\Delta l}{l} = \alpha \Delta T$ where α is coefficient of thermal expansion.

Typical values of the elastic constants are given in table below.

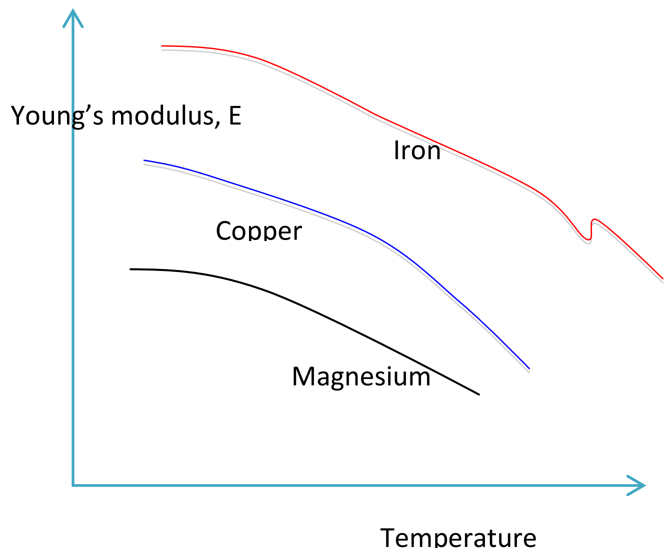


Fig. 3.2.1: Variation of Young's modulus with temperature

Table 3.2.1: Elastic properties of some metals

Material	**E (GPa)**	**Poisson's ratio**	$\alpha\ X\ 10^{-6}\ /^{o}C$
Aluminium	62	0.24	23.6
Copper	128	0.35	16.5
Iron	208	0.29	11.8
MgO	205		9

1.3 Hooke's law for tri-axial state of stress:

In a majority of forming processes, the state of stress encountered by the work piece is triaxial. Some metal forming processes such as rolling, drawing, the state of stress may be biaxial. Both tensile and compressive stresses can occur; both normal and shear stresses can co-exist. In forming processes such as extrusion and deep drawing, state of stress is triaxial.

Suppose the state of stress acting on an element of a body is plane stress, the Hooke's law for such plane stress can be written as followed:

The normal stress **σ_x** causes a normal strain which is equal to **σ_x/E.**

The normal stress acting along y direction causes a lateral strain along x direction, which is given by $-\frac{\upsilon\sigma_y}{E}$

For plane stress the total elastic strain of a body along x direction is: $\varepsilon_x = \frac{1}{E}(\sigma_x - \upsilon\sigma_y)$

Similarly, for y direction: $\varepsilon_y = \frac{1}{E}(\sigma_y - \upsilon\sigma_x)$

The z direction normal strain is now written as:

$\varepsilon_z = -\frac{\nu}{E}\left(\sigma_x + \sigma_y\right)$ This strain in z direction is due to x-direction and y-direction stresses.

For pure shear we write the Hooke's law as:

$\gamma_{xy} = \tau_{xy} / G$

Note: The normal stresses have no effect on shear strain.

Hooke's law can now be written from the above relations:

$$\sigma_x = \frac{E}{1-\nu^2}\,(\varepsilon_x + \nu\varepsilon_y)$$

$$\sigma_y = \frac{E}{1-\nu^2}\,(\varepsilon_y + \nu\varepsilon_x)$$

For triaxial state of stress:

$$\varepsilon_1 = \frac{1}{E}\,[\sigma_1 - \nu(\sigma_2 + \sigma_3)]$$

$$\varepsilon_2 = \frac{1}{E}\,[\sigma_2 - \nu(\sigma_1 + \sigma_3)]$$

$$\varepsilon_3 = \frac{1}{E}\,[\sigma_3 - \nu(\sigma_1 + \sigma_2)]$$

Solving the above equations simultaneously, we get the following stress-strain relations:

$$\sigma_1 = \frac{E}{(1+\upsilon)(1-2\upsilon)}[(1-\upsilon)\varepsilon_1 + \upsilon(\varepsilon_2 + \varepsilon_3)]$$

Similarly the relations for the other two directions can be written.

Volume change or dilatation is defined as change in volume / original volume → e = $\Delta V/V_o$

It can be shown that $$e = \varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{(1-2\nu)}{E} (\sigma_x + \sigma_y + \sigma_z)$$

1.4 Spherical stress:

If the normal stresses acting in all three directions are equal, the state of stress is called spherical stress.

Spherical stress is given by: $\sigma_o = \sigma_x = \sigma_y = \sigma_z$

The corresponding strain is given as: $\varepsilon_o = \frac{\sigma_o}{E}(1-2\nu)$

Due to spherical stress, a cube will expand or contract in size proportionately.

If the spherical stress is compressive, it is called hydrostatic stress.

Note: Hydrostatic stress cannot cause plastic deformation.

For spherical stress, we have $\sigma_x = \sigma_y = \sigma_z = \sigma_o$ and $\varepsilon_x = \varepsilon_y = \varepsilon_z = \varepsilon_o$

Therefore, e = $3\sigma_o$= $3\varepsilon_o$= $3\sigma_o \frac{(1-2\nu)}{E}$

For plastic deformation, the volumetric strain = 0 because Poisson's ratio = 0.5

Under elastic deformation, e can be >0 or e can be < 0.

1.5 Elastic Strain energy:

Strain energy is the energy stored in a material during its elastic deformation. Elastic strain energyis taken to be equal to the work done on the material during elastic deformation. Elastic wok done is equal to the area of elastic portion of stress-strain curve.

U = 1/2Fδ = ½($\sigma A \epsilon dx$) = ½($\sigma\epsilon$) $Volume$

For uniaxial stress, therefore

u = $\frac{U}{Volume} = \frac{\sigma\varepsilon}{2}$

For plane stress condition we can write the elastic strain energy per unit volume due to normal stress, u as:

$$u_1 = \frac{1}{2}\left(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y\right)$$

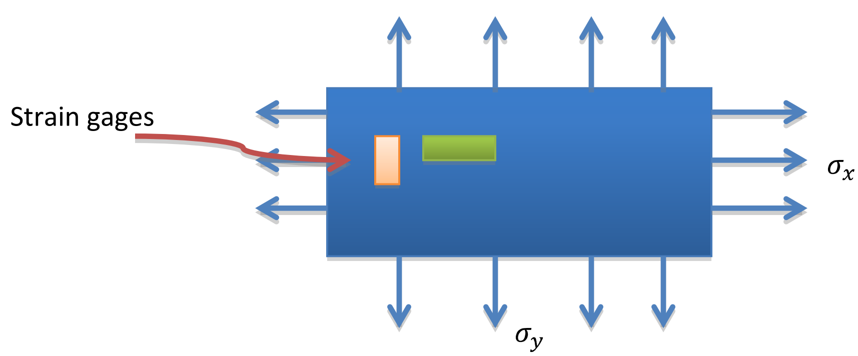
That due to shear is:

$$u_2 = \frac{\tau_{xy}\gamma_{xy}}{2}$$

Total strain energy in plane stress

u = $u_1 + u_2$

Example: A steel plate of rectangular shape with thickness t = 6 mm is subjected to normal tensile stresses along x and y directions. The two strain gages attached on the plate, one in x direction and another in y direction, give the strains as: ε_x= 0.001 and ε_x= -0.0006. Determine the two stresses and the change in thickness of the plate. Assume suitable value of E and Poisson ratio for steel.



Assume E = 200 GPa and **v = 0.3**

The state of stress given is plane stress.

Therefore we can use the stress – strain relations for plane stress:

$$\sigma_x = \frac{E}{1-\nu^2}\,(\varepsilon_x + \nu\varepsilon_y)$$

Substituting the values of the strains and the other parameters, we get the stress:

σ_x= 200X10^3(0.001-0.3X0.0006)/(1-0.3^2) = 180.22MPa

Similarly,

σ_y= -65.93 MPa. (Compressive)

Now we have $\varepsilon_z = -\frac{v}{E}(\sigma_x + \sigma_y)$ = -1.71X10^{-4}, This strain is contraction strain. There is reduction in thickness.

Source:
http://nptel.ac.in/courses/112106153/8