Elastic Strain Energy

The strain energy stored in an elastic material upon deformation is calculated below for a number of different geometries and loading conditions. These expressions for stored energy will then be used to solve some elasticity problems using the energy methods mentioned in the previous section.

8.2.1 Strain energy in deformed Components

Bar under axial load

Consider a bar of elastic material fixed at one end and subjected to a steadily increasing force *P*, Fig. 8.2.1. The force is applied slowly so that kinetic energies are negligible. The initial length of the bar is L. The work dW done in extending the bar a small amount $d\Delta$ is[1]

$$dW = Pd\Delta \tag{8.2.1}$$

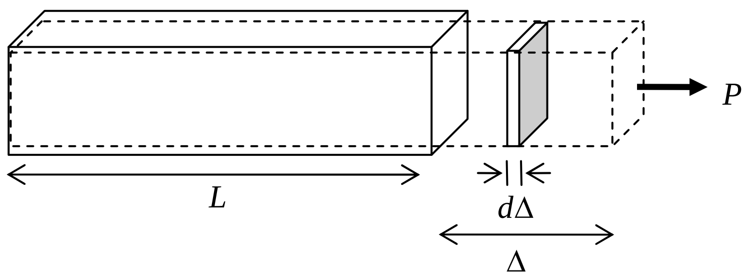

Figure 8.2.1: a bar loaded by a force

It was shown in §7.1.2 that the force and extension Δ are linearly related through $\Delta = PL/EA$, Eqn. 7.1.5, where *E* is the Young's modulus and *A* is the cross sectional area. This linear relationship is plotted in Fig. 8.2.2. The work expressed by Eqn. 8.2.1 is the white region under the force-extension curve (line). The total work done during the complete extension up to a *final* force *P* and *final* extension Δ is the total area beneath the curve.

The work done is stored as elastic strain energy *U* and so

$$U = \frac{1}{2}P\Delta = \frac{P^2L}{2EA} \tag{8.2.2}$$

If the axial force (and/or the cross-sectional area and Young's modulus) varies along the bar, then the above calculation can be done for a small element of length dx. The energy stored in this element would be $P^2dx/2EA$ and the total strain energy stored in the bar would be

[1] the small change in force during this small extension may be neglected

$$U = \int_0^L \frac{P^2}{2EA} dx \tag{8.2.3}$$

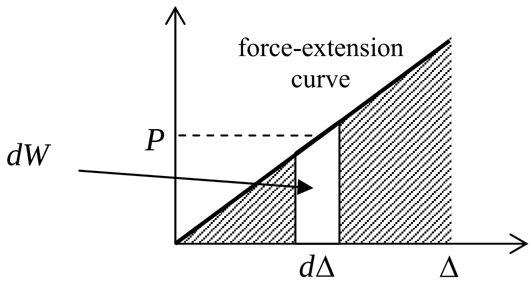

Figure 8.2.2: force-displacement curve for uniaxial load

The strain energy is always positive, due to the square on the force *P*, regardless of whether the bar is being compressed or elongated.

Note the factor of one half in Eqn. 8.2.2. The energy stored is not simply force times displacement because *the force is changing* during the deformation.

Circular Bar in Torsion

Consider a circular bar subjected to a torque *T*. The torque is equivalent to a couple: two forces of magnitude *F* acting in opposite directions and separated by a distance $2r$ as in Fig. 8.2.3; $T = 2Fr$. As the bar twists through a small angle $\Delta\phi$, the forces each move through a distance $\Delta s = r\Delta\phi$. The work done is therefore $\Delta W = 2(F\Delta s) = T\Delta\phi$.

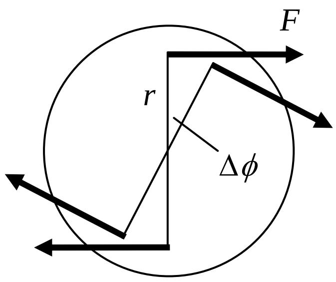

Figure 8.2.3: torque acting on a circular bar

It was shown in §7.2 that the torque and angle of twist are linearly related through Eqn. 7.2.10, $\phi = TL/GJ$, where *L* is the length of the bar, *G* is the shear modulus and *J* is the polar moment of inertia. The angle of twist can be plotted against the torque as in Fig. 8.2.4.

The total strain energy stored in the cylinder during the straining up to a final angle of twist ϕ is the work done, equal to the shaded area in Fig. 8.2.4, leading to

$$U = \frac{1}{2}\phi T = \frac{T^2 L}{2GJ} \tag{8.2.4}$$

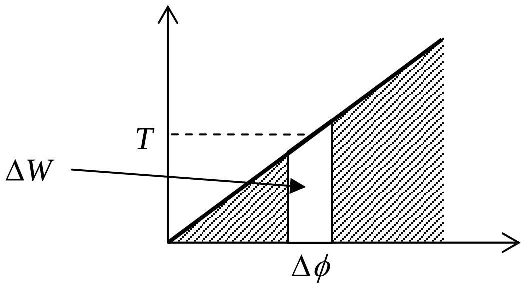

Figure 8.2.4: torque – angle of twist plot for torsion

Again, if the various quantities are varying along the length of the bar, then the total strain energy can be expressed as

$$U = \int_0^L \frac{T^2}{2GJ} dx \qquad (8.2.5)$$

Beam subjected to a Pure Moment

As with the bar under torsion, the work done by a moment M as it moves through an angle $d\theta$ is $Md\theta$. The moment is related to the radius of curvature R through Eqns. 7.4.36-37, $M = EI / R$, where E is the Young's modulus and I is the moment of inertia. The length L of a beam and the angle subtended θ are related to R through $L = R\theta$, Fig. 8.2.5, and so moment and angle θ are linearly related through $\theta = ML / EI$.

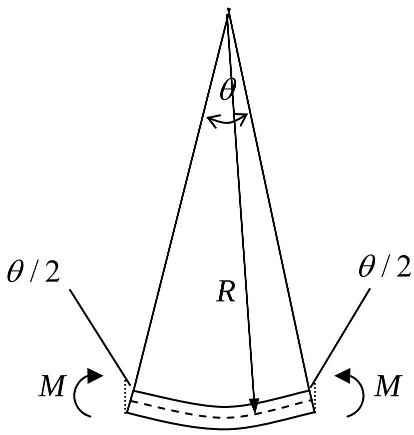

Figure 8.2.5: beam of length *L* under pure bending

The total strain energy stored in a bending beam is then

$$U = \frac{1}{2}\theta M = \frac{M^2 L}{2EI} \qquad (8.2.6)$$

and if the moment and other quantities vary along the beam,

$$U = \int_0^L \frac{M^2}{2EI} dx \tag{8.2.7}$$

This expression is due to the flexural stress σ. A beam can also store energy due to shear stress τ; this latter energy is usually much less than that due to the flexural stresses provided the beam is slender – this is discussed further below.

Example

Consider the bar with varying circular cross-section shown in Fig. 8.2.6. The Young's modulus is 200GPa.

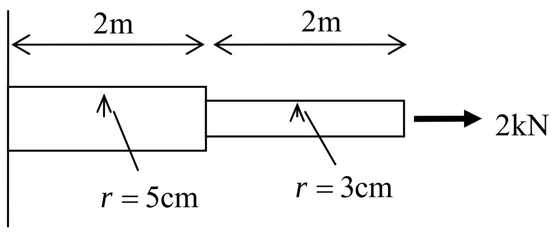

Figure 8.2.6: a loaded bar

The strain energy stored in the bar when a force of 2kN is applied at the free end is

$$U = \int_0^L \frac{P^2}{2EA} dx = \frac{(2\times 10^3)^2 (2)}{2(2\times 10^{11})\pi}\left(\frac{1}{(5\times 10^{-2})^2} + \frac{1}{(3\times 10^{-2})^2}\right) = 9.62\times 10^{-3}\ \text{Nm} \tag{8.2.8}$$

■

8.2.2 The Work-Energy Principle

The work-energy principle for elastic materials, that is, the fact that the work done by external forces is stored as elastic energy, can be used directly to solve some simple problems. To be precise, it can be used to solve problems involving a single force and for solving for the displacement in the direction of that force. By force and displacement here it is meant **generalised force** and **generalised displacement**, that is, a force/displacement pair, a torque/angle of twist pair or a moment/bending angle pair.

More complex problems need to be solved using more sophisticated energy methods, such as Castiglianos' method discussed further below.

Example

Consider the beam of length *L* shown in Fig. 8.2.7, pinned at one end (A) and simply supported at the other (C). A moment M_0 acts at B, a distance L_1 from the left-hand end. The cross-section is rectangular with depth *b* and height *h*. The work-energy principle can be used to calculate the angle θ_B through which the moment at B rotates.

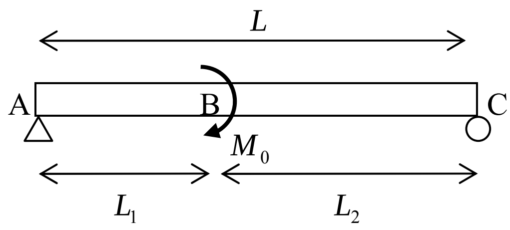

Figure 8.2.7: a beam subjected to a moment at B

The moment along the beam can be calculated from force and moment equilibrium,

$$M = \begin{cases} -M_0 x/L, & 0 < x < L_1 \\ M_0(1 - x/L), & L_1 < x < L \end{cases} \tag{8.2.9}$$

The strain energy stored in the bar (due to the flexural stresses only) is

$$U = \int_0^L \frac{M^2}{2EI} dx = \frac{6M_0^2}{Ebh^3}\left\{ \int_0^{L_1} \left(\frac{x}{L}\right)^2 dx + \int_{L_1}^{L_2} \left(1 - \frac{x}{L}\right)^2 dx \right\} = \frac{2M_0^2 L_2^3}{Ebh^3 L^2} \tag{8.2.10}$$

The work done by the applied moment is $M_0\theta_B / 2$ and so

$$\theta_B = \frac{4M_0 L_2^3}{Ebh^3 L^2} \tag{8.2.11}$$

■

8.2.3 Strain Energy Density

The strain energy will in general vary throughout a body and for this reason it is useful to introduce the concept of **strain energy density**, which is a measure of how much energy is stored in small volume elements throughout a material.

Consider again a bar subjected to a uniaxial force *P*. A small volume element with edges aligned with the x, y, z axes as shown in Fig. 8.2.8 will then be subjected to a stress σ_{xx} only. The volume of the element is $dV = dxdydz$.

From Eqn. 8.2.2, the strain energy in the element is

$$U = \frac{(\sigma_{xx} dydz)^2 dx}{2Edydz} \tag{8.2.12}$$

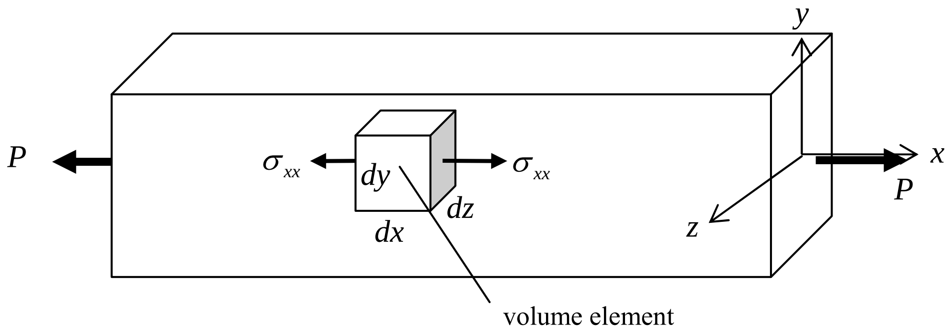

Figure 8.2.8: a volume element under stress

The strain energy density *u* is defined as the strain energy *per unit volume*:

$$u = \frac{\sigma_{xx}^2}{2E} \tag{8.2.13}$$

The total strain energy in the bar may now be expressed as this quantity integrated over the whole volume,

$$U = \int_V u dV , \tag{8.2.14}$$

which, for a constant cross-section *A* and length *L* reads $U = A\int_0^L u dx$. From Hooke's law, the strain energy density of Eqn. 8.2.13 can also be expressed as

$$u = \frac{1}{2}\sigma_{xx}\varepsilon_{xx} \tag{8.2.15}$$

As can be seen from Fig. 8.2.9, this is the area under the uniaxial stress-strain curve.

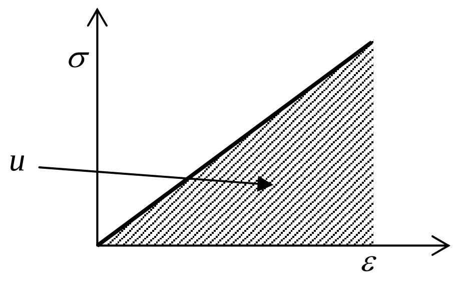

Figure 8.2.9: stress-strain curve for elastic material

Note that the element *does* deform in the *y* and *z* directions but no work is associated with those displacements since there is no force acting in those directions.

The strain energy density for an element subjected to a σ_{yy} stress only is, by the same arguments, $\sigma_{yy}\varepsilon_{yy}/2$, and that due to a σ_{zz} stress is $\sigma_{zz}\varepsilon_{zz}/2$. Consider next a shear stress σ_{xy} acting on the volume element to produce a shear strain ε_{xy} as illustrated in Fig. 8.2.10. The element deforms with small angles θ and λ as illustrated. Only the stresses on the upper and right-hand surfaces are shown, since the stresses on the other two surfaces do no work. The force acting on the upper surface is $\sigma_{xy}dxdz$ and moves through a displacement λdy. The force acting on the right-hand surface is $\sigma_{xy}dydz$ and moves through a displacement θdx. The work done when the element moves through angles $d\theta$ and $d\lambda$ is then, using the definition of shear strain,

$$dW = (\sigma_{xy}dxdz)(d\lambda dy) + (\sigma_{xy}dydz)(d\theta dx) = (dxdydz)\sigma_{xy}(2d\varepsilon_{xy}) \qquad (8.2.16)$$

and, with shear stress proportional to shear strain, the strain energy density is

$$u = 2\int \sigma_{xy}d\varepsilon_{xy} = \sigma_{xy}\varepsilon_{xy} \qquad (8.2.17)$$

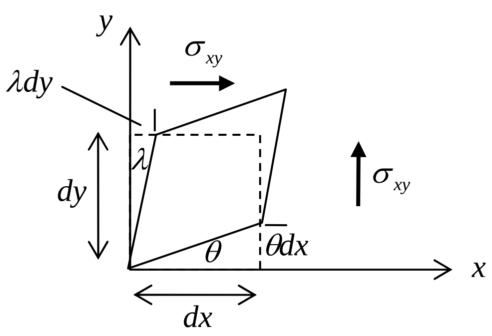

Figure 8.2.10: a volume element under shear stress

The strain energy can be similarly calculated for the other shear stresses and, in summary, the strain energy density for a volume element subjected to arbitrary stresses is

$$u = \frac{1}{2}(\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz}) + (\sigma_{xy}\varepsilon_{xy} + \sigma_{yz}\varepsilon_{yz} + \sigma_{zx}\varepsilon_{zx}) \qquad (8.2.18)$$

Using Hooke's law, Eqns. 6.1.9, and Eqn. 6.1.5, the strain energy density can also be written in the alternative and useful forms {▲Problem 4}

$$\begin{aligned} u &= \frac{1}{2E}(\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2) - \frac{\nu}{E}(\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx}) + \frac{1}{2\mu}(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2) \\ &= \frac{\mu}{1-2\nu}\left[(1-\nu)(\varepsilon_{xx}^2 + \varepsilon_{yy}^2 + \varepsilon_{zz}^2) + 2\nu(\varepsilon_{xx}\varepsilon_{yy} + \varepsilon_{yy}\varepsilon_{zz} + \varepsilon_{zz}\varepsilon_{xx})\right] + 2\mu(\varepsilon_{xy}^2 + \varepsilon_{yz}^2 + \varepsilon_{zx}^2) \\ &= \frac{\nu\mu}{1-2\nu}(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz})^2 + \mu(\varepsilon_{xx}^2 + \varepsilon_{yy}^2 + \varepsilon_{zz}^2) + 2\mu(\varepsilon_{xy}^2 + \varepsilon_{yz}^2 + \varepsilon_{zx}^2) \end{aligned}$$

(8.2.19)

Strain Energy in a Beam due to Shear Stress

The shear stresses arising in a beam at location *y* from the neutral axis are given by Eqn. 7.4.28, $\tau(y) = Q(y)V / Ib(y)$, where *Q* is the first moment of area of the section of beam from *y* to the outer surface, *V* is the shear force, *I* is the moment of inertia of the complete cross-section and *b* is the thickness of the beam at *y*. From Eqns. 8.2.19a and 8.2.14 then, the total strain energy in a beam of length *L* due to shear stress is

$$U = \int_V \frac{\tau^2}{2\mu} dV = \frac{1}{2}\int_0^L \frac{V^2}{\mu I^2}\left[\int_A \frac{Q^2}{b^2} dA\right] dx \qquad (8.2.20)$$

Here V, μ and *I* are taken to be constant for any given cross-section but may vary along the beam; Q varies and *b* may vary over any given cross-section. Expression 8.2.20 can be simplified by introducing the **form factor for shear** f_s, defined as

$$f_s(x) = \frac{A}{I^2}\int_A \frac{Q^2}{b^2} dA \qquad (8.2.21)$$

so that

$$U = \frac{1}{2}\int_0^L \frac{f_s V^2}{\mu A} dx \qquad (8.2.22)$$

The form factor depends only on the shape of the cross-section. For example, for a rectangular cross-section, using Eqn. 7.4.29,

$$f_s(x) = \frac{bh}{\left(bh^3/12\right)^2}\int_{-h/2}^{+h/2} \frac{1}{b^2}\left[\frac{b}{2}\left(\frac{h^2}{4} - y^2\right)\right]^2 dy \int_{-b/2}^{+b/2} dz = \frac{6}{5} \qquad (8.2.23)$$

In a similar manner, the form factor for a circular cross-section is found to be $10/9$ and that of a very thin tube is 2.

8.2.4 Castigliano's Second Theorem

The work-energy method is the simplest of energy methods. A more powerful method is that based on **Castigliano's second theorem**[2], which can be used to solve problems involving *linear* elastic materials. As an introduction to Castigliano's second theorem, consider the case of uniaxial tension, where $U = P^2 L / 2EA$. The displacement through which the force moves can be obtained by a differentiation of this expression with respect to that force,

$$\frac{dU}{dP} = \frac{PL}{EA} = \Delta \qquad (8.2.24)$$

[2] Casigliano's first theorem will be discussed in a later section

Similarly, for torsion of a circular bar, $U = T^2 L / 2GJ$, and a differentiation gives $dU / dT = TL / GJ = \phi$. Further, for bending of a beam it is also seen that $dU / dM = \theta$.

These are examples of Castigliano's theorem, which states that, provided the body is in equilibrium, *the derivative of the strain energy with respect to the force gives the displacement corresponding to that force, in the direction of that force*. When there is more than one force applied, then one takes the partial derivative. For example, if *n* independent forces $P_1, P_2, \ldots, P_n$ act on a body, the displacement corresponding to the *i*th force is

$$\Delta_i = \frac{\partial U}{\partial P_i} \tag{8.2.25}$$

Before proving this theorem, here follow some examples.

Example

The beam shown in Fig. 8.2.11 is pinned at A, simply supported half-way along the beam at B and loaded at the end C by a force *P* and a moment M_0.

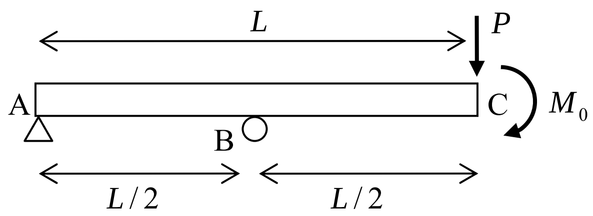

Figure 8.2.11: a beam subjected to a force and moment at C

The moment along the beam can be calculated from force and moment equilibrium,

$$M = \begin{cases} -Px - 2M_0 x / L, & 0 < x < L/2 \\ -M_0 - P(L - x), & L/2 < x < L \end{cases} \tag{8.2.26}$$

The strain energy stored in the bar (due to the flexural stresses only) is

$$\begin{aligned} U &= \frac{1}{2EI}\left\{\left(P + \frac{2M_0}{L}\right)^2 \int_0^{L/2} x^2 dx + \int_{L/2}^{L} \left(M_0 + P(L - x)\right)^2 dx\right\} \\ &= \frac{P^2 L^3}{24EI} + \frac{5PM_0 L^2}{24EI} + \frac{M_0^2 L}{3EI} \end{aligned} \tag{8.2.27}$$

In order to apply Castigliano's theorem, the strain energy is considered to be a function of the two external loads, $U = U(P, M_0)$. The displacement associated with the force *P* is then

$$\Delta_C = \frac{\partial U}{\partial P} = \frac{PL^3}{12EI} + \frac{5M_0 L^2}{24EI} \tag{8.2.28}$$

The rotation associated with the moment is

$$\theta_C = \frac{\partial U}{\partial M_0} = \frac{5PL^2}{24EI} + \frac{2M_0 L}{3EI} \tag{8.2.29}$$

■

Example

Consider next the beam of length *L* shown in Fig. 8.2.12, built in at both ends and loaded centrally by a force *P*. This is a statically indeterminate problem. In this case, the strain energy can be written as a function of the applied load and one of the unknown reactions.

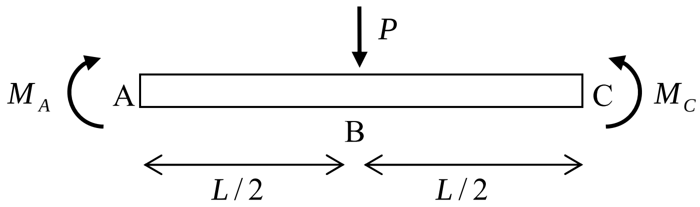

Figure 8.2.12: a statically indeterminate beam

First, the moment in the beam is found from equilibrium considerations to be

$$M = M_A + \frac{P}{2}x, \qquad 0 < x < L/2 \tag{8.2.30}$$

where M_A is the unknown reaction at the left-hand end. Then the strain energy in the left-hand half of the beam is

$$U = \frac{1}{2EI}\int_0^{L/2}\left(M_A + \frac{P}{2}x\right)^2 dx = \frac{P^2L^3}{192EI} + \frac{PM_A L^2}{16EI} + \frac{M_A^2 L}{4EI} \tag{8.2.31}$$

The strain energy in the complete beam is double this:

$$U = \frac{P^2L^3}{96EI} + \frac{PM_A L^2}{8EI} + \frac{M_A^2 L}{2EI} \tag{8.2.32}$$

Writing the strain energy as $U = U(P, M_A)$, the rotation at A is

$$\theta_A = \frac{\partial U}{\partial M_A} = \frac{PL^2}{8EI} + \frac{M_A L}{EI} \tag{8.2.33}$$

But $\theta_A = 0$ and so Eqn. 8.2.33 can be solved to get $M_A = -PL/8$. Then the displacement at the centre of the beam is

$$\Delta_B = \frac{\partial U}{\partial P} = \frac{PL^3}{48EI} + \frac{M_A L^2}{8EI} = \frac{PL^3}{192EI} \qquad (8.2.34)$$

This is positive in the direction in which the associated force is acting, and so is downward.

■

Proof of Castigliano's Theorem

A proof of Castiligliano's theorem will be given here for a structure subjected to a single load. The load P produces a displacement Δ and the strain energy is $U = P\Delta/2$, Fig. 8.2.13. If an additional force dP is applied giving an additional deformation $d\Delta$, the additional strain energy is

$$dU = Pd\Delta + \frac{1}{2}dPd\Delta \qquad (8.2.35)$$

If the load $P + dP$ is applied in one step, the work done is $(P + dP)(\Delta + d\Delta)/2$. Equating this to the strain energy $U + dU$ given by Eqn. 8.2.35 then gives $Pd\Delta = \Delta dP$. Substituting into Eqn. 8.2.35 leads to

$$dU = \Delta dP + \frac{1}{2}dPd\Delta \qquad (8.2.36)$$

Dividing through by dP and taking the limit as $d\Delta \to 0$ results in Castigliano's second theorem, $dU/dP = \Delta$.

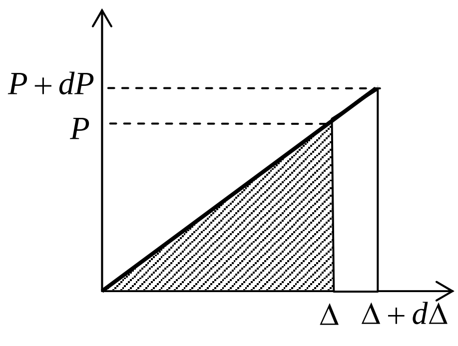

Figure 8.2.13: force-displacement curve

In fact, dividing Eqn. 8.2.35 through by $d\Delta$ and taking the limit as $d\Delta \to 0$ results in **Castigliano's first theorem**, $dU/d\Delta = P$. It will be shown later that this first theorem, unlike the second, in fact holds also for the case when the elastic material is *non-linear*.

8.2.5 Dynamic Elasticiy

Impact and Dynamic Loading

Consider the case of a weight P dropped instantaneously onto the end of an elastic bar. If the weight P had been applied gradually from zero, the strain energy stored at the final force P and final displacement Δ_0 would be $\frac{1}{2}P\Delta_0$. However, the instantaneously applied load is constant throughout the deformation and work done up to a displacement Δ_0 is $P\Delta_0$, Fig. 8.2.14. The difference between the two implies that the bar acquires a kinetic energy (see Eqn. 8.1.19); the material particles accelerate from their equilibrium positions during the compression.

As deformation proceeds beyond Δ_0, it is clear from Fig. 8.2.14 that the strain energy is increasing faster than the work being done by the weight and so there must be a drop in kinetic energy; the particles begin to decelerate. Eventually, at $\Delta_{max} = 2\Delta_0$, the work done by the weight exactly equals the strain energy stored and the material is at rest. However, the material is not in equilibrium – the equilibrium position for a load P is Δ_0 – and so the material begins to accelerate back to Δ_0.

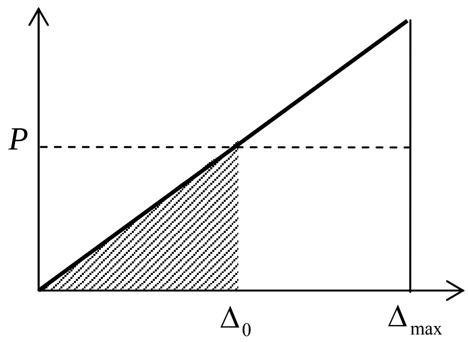

Figure 8.2.14: non-equilibrium loading

The bar and weight will continue to oscillate between 0 and Δ_{max} indefinitely. In a real (inelastic) material, internal friction will cause the vibration to decay.

Thus the maximum compression of a bar under impact loading is twice that of a bar subjected to the same load gradually.

Example

Consider a weight w dropped from a height h. If one is interested in the final, maximum, displacement of the bar, Δ_{max}, one does not need to know about the detailed and complex transfer of energies during the impact; the energy lost by the weight equals the strain energy stored in the bar:

$$w(h + \Delta_{max}) = \frac{1}{2}P\Delta_{max} \qquad (8.2.37)$$

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