

Elastic Stability Of Columns

Introduction:

Structural members which carry compressive loads may be divided into two broad categories depending on their relative lengths and cross-sectional dimensions.

Columns:

Short, thick members are generally termed columns and these usually fail by crushing when the yield stress of the material in compression is exceeded.

Struts:

Long, slender columns are generally termed as struts, they fail by buckling some time before the yield stress in compression is reached. The buckling occurs owing to one the following reasons.

- (a). the strut may not be perfectly straight initially.
- (b). the load may not be applied exactly along the axis of the Strut.
- (c). one part of the material may yield in compression more readily than others owing to some lack of uniformity in the material properties through out the strut.

In all the problems considered so far we have assumed that the deformation to be both progressive with increasing load and simple in form i.e. we assumed that a member in simple tension or compression becomes progressively longer or shorter but remains straight. Under some circumstances however, our assumptions of progressive and simple deformation may no longer hold good and the member become unstable. The term strut and column are widely used, often interchangeably in the context of buckling of slender members.]

At values of load below the buckling load a strut will be in stable equilibrium where the displacement caused by any lateral disturbance will be totally recovered when the disturbance is removed. At the buckling load the strut is said to be in a state of neutral equilibrium, and theoretically it should than be possible to gently deflect the strut into a simple sine wave provided that the amplitude of wave is kept small.

Theoretically, it is possible for struts to achieve a condition of unstable equilibrium with loads exceeding the buckling load, any slight lateral disturbance then causing failure by buckling, this condition is never achieved in practice under static load conditions. Buckling occurs immediately at the point where the buckling load is reached, owing to the reasons stated earlier.

The resistance of any member to bending is determined by its flexural rigidity EI and is The quantity I may be written as $I = Ak^2$,

Where I = area of moment of inertia

A = area of the cross-section

k = radius of gyration.

The load per unit area which the member can withstand is therefore related to k . There will be two principal moments of inertia, if the least of these is taken then the ratio

$$\frac{l}{k} \quad \text{i.e.} \quad \frac{\text{length of member}}{\text{least radius of gyration}}$$

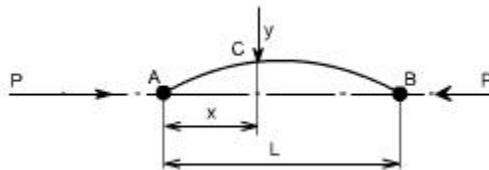
Is called the slenderness ratio. It's numerical value indicates whether the member falls into the class of columns or struts.

Euler's Theory : The struts which fail by buckling can be analyzed by Euler's theory. In the following sections, different cases of the struts have been analyzed.

Case A: Strut with pinned ends:

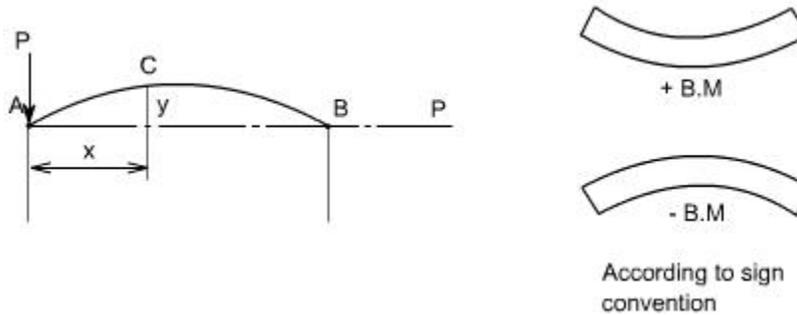
Consider an axially loaded strut, shown below, and is subjected to an axial load 'P' this load 'P' produces a deflection 'y' at a distance 'x' from one end.

Assume that the ends are either pin jointed or rounded so that there is no moment at either end.



Assumption:

The strut is assumed to be initially straight, the end load being applied axially through centroid.



$$B. M|_C = -Py$$

Further, we know that

$$EI \frac{d^2 y}{dx^2} = M$$

$$EI \frac{d^2 y}{dx^2} = - P \cdot y = M$$

In this equation 'M' is not a function 'x'. Therefore this equation can not be integrated directly as has been done in the case of deflection of beams by integration method.

Thus,

$$EI \frac{d^2 y}{dx^2} + P y = 0$$

Though this equation is in 'y' but we can't say at this stage where the deflection would be maximum or minimum.

$$\frac{d^2 y}{dx^2} + \frac{Py}{EI} = 0$$

So the above differential equation can be arranged in the following form

Let us define a operator

$$D = d/dx$$

$$(D^2 + n^2) y = 0 \text{ where } n^2 = P/EI$$

This is a second order differential equation which has a solution of the form consisting of complimentary function and particular integral but for the time being we are interested in the complementary solution only [in this P.I = 0; since the R.H.S of Diff. equation = 0]

$$\text{Thus } y = A \cos (nx) + B \sin (nx)$$

Where A and B are some constants.

Therefore
$$y = A \cos \sqrt{\frac{P}{EI}} x + B \sin \sqrt{\frac{P}{EI}} x$$

In order to evaluate the constants A and B let us apply the boundary conditions,

(i) at $x = 0$; $y = 0$

(ii) at $x = L$; $y = 0$

Applying the first boundary condition yields $A = 0$.

Applying the second boundary condition gives

$$B \sin \left(L \sqrt{\frac{P}{EI}} \right) = 0$$

Thus either $B = 0$, or $\sin \left(L \sqrt{\frac{P}{EI}} \right) = 0$

if $B=0$, that $y=0$ for all values of x hence the strut has not buckled yet. Therefore, the solution required is

$$\sin \left(L \sqrt{\frac{P}{EI}} \right) = 0 \text{ or } \left(L \sqrt{\frac{P}{EI}} \right) = \pi \text{ or } nL = \pi$$

$$\text{or } \sqrt{\frac{P}{EI}} = \frac{\pi}{L} \text{ or } P = \frac{\pi^2 EI}{L^2}$$

From the above relationship the least value of P which will cause the strut to buckle, and it is called the “ **Euler Crippling Load** ” P_e from which we obtain.

$$P_e = \frac{\pi^2 EI}{L^2}$$

It may be noted that the value of I used in this expression is the least moment of inertia

It should be noted that the other solutions exists for the equation

$$\sin \left(L \sqrt{\frac{P}{EI}} \right) = 0 \quad \text{i.e. } \sin nL = 0$$

The interpretation of the above analysis is that for all the values of the load P , other than those which make $\sin nL = 0$; the strut will remain perfectly straight since

$$y = B \sin nL = 0$$

For the particular value of

$$P_e = \frac{\pi^2 EI}{L^2}$$

$$\sin nL = 0 \quad \text{or } nL = \pi$$

$$\text{Therefore } n = \frac{\pi}{L}$$

$$\text{Hence } y = B \sin nx = B \sin \frac{\pi x}{L}$$

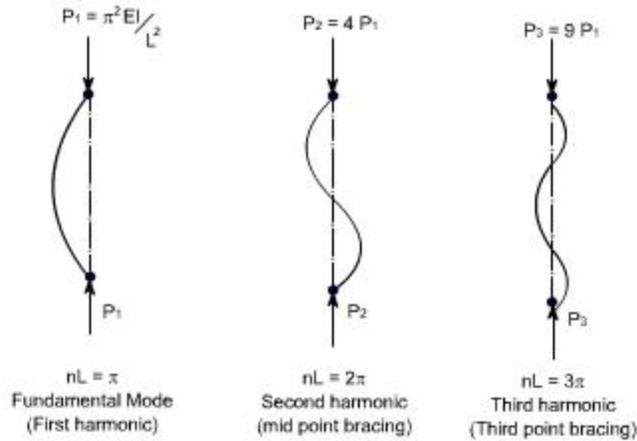
Then we say that the strut is in a state of neutral equilibrium, and theoretically any deflection which it suffers will be maintained. This is subjected to the limitation that 'L' remains sensibly constant and in practice slight increase in load at the critical value will cause the deflection to increase appreciably until the material fails by yielding.

Further it should be noted that the deflection is not proportional to load, and this applies to all strut problems; like wise it will be found that the maximum stress is not proportional to load.

The solution chosen of $nL = \pi$ is just one particular solution; the solutions $nL = 2\pi, 3\pi, 5\pi$ etc are equally valid mathematically and they do, in fact, produce values of ' P_e ' which are equally valid for modes of buckling of strut different from that of a simple bow. Theoretically therefore, there are an infinite number of values of P_e , each corresponding with a different mode of buckling.

The value selected above is so called the fundamental mode value and is the lowest critical load producing the single bow buckling condition.

The solution $nL = 2\pi$ produces buckling in two half – waves, 3π in three half-waves etc.



$$L\sqrt{\frac{P}{EI}} = \pi \text{ or } P_1 = \frac{\pi^2 EI}{L^2}$$

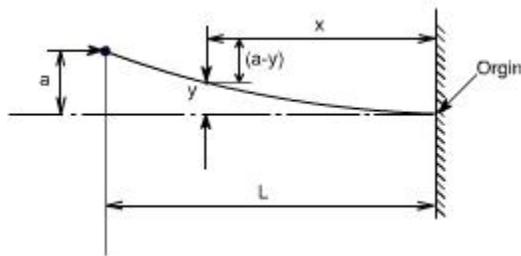
$$\text{If } L\sqrt{\frac{P}{EI}} = 2\pi \text{ or } P_2 = \frac{4\pi^2 EI}{L^2} = 4P_1$$

$$\text{If } L\sqrt{\frac{P}{EI}} = 3\pi \text{ or } P_3 = \frac{9\pi^2 EI}{L^2} = 9P_1$$

If load is applied sufficiently quickly to the strut, then it is possible to pass through the fundamental mode and to achieve at least one of the other modes which are theoretically possible. In practical loading situations, however, this is rarely achieved since the high stress associated with the first critical condition generally ensures immediate collapse.

struts and columns with other end conditions: Let us consider the struts and columns having different end conditions

Case b: One end fixed and the other free:



writing down the value of bending moment at the point C

$$B. M|_c = P(a - y)$$

Hence, the differential equation becomes,

$$EI \frac{d^2 y}{dx^2} = P(a - y)$$

On rearranging we get

$$\frac{d^2 y}{dx^2} + \frac{Py}{EI} = \frac{Pa}{EI}$$

$$\text{Let } \frac{P}{EI} = n^2$$

Hence in operator form, the differential equation reduces to $(D^2 + n^2)y = n^2 a$

The solution of the above equation would consist of complementary solution and particular solution, therefore

$$y_{\text{gen}} = A \cos(nx) + \sin(nx) + P.I$$

where

P.I = the P.I is a particular value of y which satisfies the differential equation

$$\text{Hence } y_{P.I} = a$$

Therefore the complete solution becomes

$$Y = A \cos(nx) + B \sin(nx) + a$$

Now imposing the boundary conditions to evaluate the constants A and B

$$(i) \text{ at } x = 0; y = 0$$

$$\text{This yields } A = -a$$

$$(ii) \text{ at } x = 0; dy/dx = 0$$

$$\text{This yields } B = 0$$

Hence

$$y = -a \cos(nx) + a$$

$$\text{Further, at } x = L; y = a$$

$$\text{Therefore } a = -a \cos(nL) + a \quad \text{or } 0 = \cos(nL)$$

Now the fundamental mode of buckling in this case would be

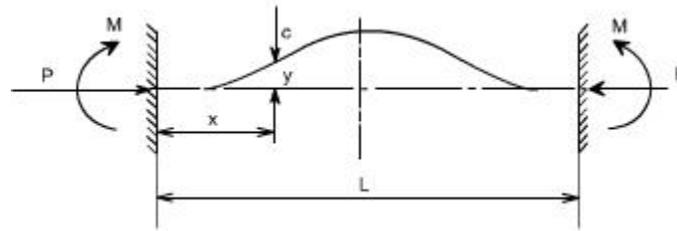
$$nL = \frac{\pi}{2}$$

$$\sqrt{\frac{P}{EI}} L = \frac{\pi}{2}, \text{ Therefore, the Euler's crippling load is given as}$$

$$P_e = \frac{\pi^2 EI}{4L^2}$$

Case 3

Strut with fixed ends:



Due to the fixed end supports bending moment would also appears at the supports, since this is the property of the support.

Bending Moment at point C = $M - P \cdot y$

$$EI \frac{d^2 y}{dx^2} = M - Py$$

$$\text{or } \frac{d^2 y}{dx^2} + \frac{P}{EI} = \frac{M}{EI}$$

$n^2 = \frac{P}{EI}$, Therefore in the operator form, the equation reduces to

$$(D^2 + n^2)y = \frac{M}{EI}$$

$y_{\text{general}} = y_{\text{complementary}} + y_{\text{particular integral}}$

$$y|_{\text{P.I}} = \frac{M}{n^2 EI} = \frac{M}{P}$$

Hence the general solution would be

$$y = B \cos nx + A \sin nx + \frac{M}{P}$$

Boundary conditions relevant to this case are at $x=0; y=0$

$$B = -\frac{M}{P}$$

Also at $x=0; \frac{dy}{dx} = 0$ hence

$$A=0$$

Therefore,

$$y = -\frac{M}{P} \cos nx + \frac{M}{P}$$

$$y = \frac{M}{P} (1 - \cos nx)$$

Further, it may be noted that at $x=L; y=0$

$$\text{Then } 0 = \frac{M}{P} (1 - \cos nL)$$

Thus, either $\frac{M}{P} = 0$ or $(1 - \cos nL) = 0$

obviously, $(1 - \cos nL) = 0$

$$\cos nL = 1$$

Hence the least solution would be

$$nL = 2\pi$$

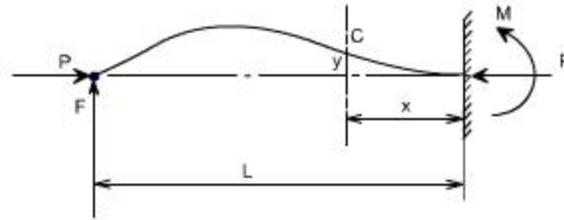
$\sqrt{\frac{P}{EI}} L = 2\pi$, Thus, the buckling load or crippling load is

$$P_e = \frac{4\pi^2 \cdot EI}{L^2}$$

Thus,

Case 4

One end fixed, the other pinned



In order to maintain the pin-joint on the horizontal axis of the unloaded strut, it is necessary in this case to introduce a vertical load F at the pin. The moment of F about the built in end then balances the fixing moment.

With the origin at the built in end, the B,M at C is given as

$$EI \frac{d^2 y}{dx^2} = -Py + F(L - x)$$

$$EI \frac{d^2 y}{dx^2} + Py = F(L - x)$$

Hence

$$\frac{d^2 y}{dx^2} + \frac{P}{EI} y = \frac{F}{EI} (L - x)$$

In the operator form the equation reduces to

$$(D^2 + n^2) y = \frac{F}{EI} (L - x)$$

$$y_{\text{particular}} = \frac{F}{n^2 EI} (L - x) \text{ or } y = \frac{F}{P} (L - x)$$

The full solution is therefore

$$y = A \cos nx + B \sin nx + \frac{F}{P} (L - x)$$

The boundry conditions relevant to the problem are at $x=0; y=0$

$$\text{Hence } A = -\frac{FL}{P}$$

$$\text{Also at } x=0; \frac{dy}{dx} = 0$$

$$\text{Hence } B = \frac{F}{nP}$$

$$\text{or } y = -\frac{FL}{P} \cos nx + \frac{F}{nP} \sin nx + \frac{F}{P} (L - x)$$

$$y = \frac{F}{nP} [\sin nx - nL \cos nx + n(L - x)]$$

Also when $x = L; y = 0$

Therefore

$$nL \cos nL = \sin nL \quad \text{or } \tan nL = nL$$

The lowest value of nL (neglecting zero) which satisfies this condition and which therefore produces the fundamental buckling condition is $nL = 4.49$ radian

$$\text{or } \sqrt{\frac{P}{EI}} L = 4.49$$

$$\frac{P_e L^2}{EI} = 20.2$$

$$P_e = \frac{2.05\pi^2 EI}{L^2}$$

Equivalent Strut Length:

Having derived the results for the buckling load of a strut with pinned ends the Euler loads for other end conditions may all be written in the same form.

$$\text{i.e. } P_e = \frac{\pi^2 EI}{L^2}$$

Where L is the equivalent length of the strut and can be related to the actual length of the strut depending on the end conditions.

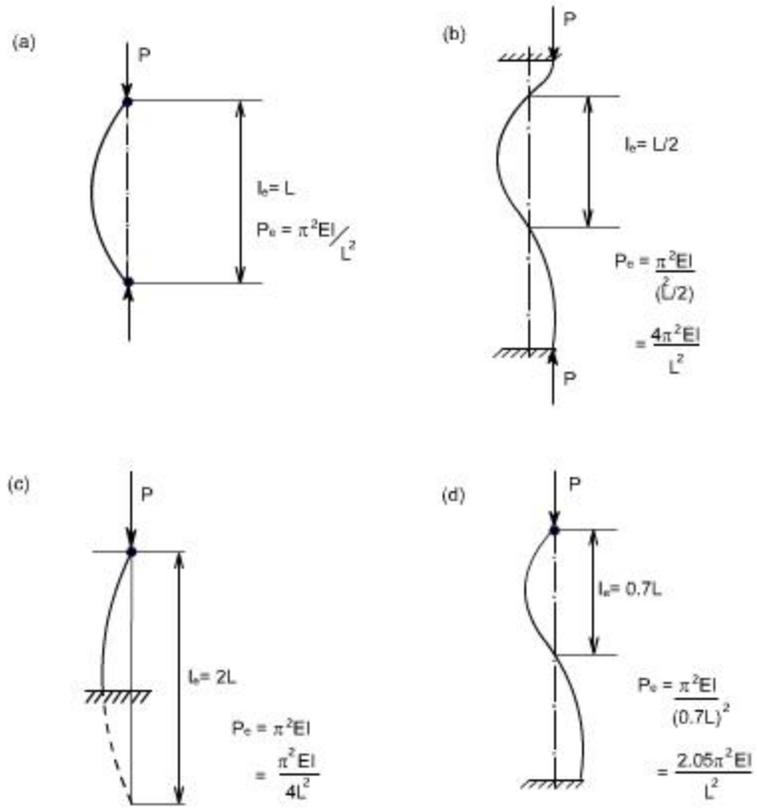
The equivalent length is found to be the length of a simple bow(half sine wave) in each of the strut deflection curves shown. The buckling load for each end condition shown is then readily obtained. The use of equivalent length is not restricted to the Euler's theory and it will be used in other derivations later.

The critical load for columns with other end conditions can be expressed in terms of the critical load for a hinged column, which is taken as a fundamental case.

For case(c) see the figure, the column or strut has inflection points at quarter points of its unsupported length. Since the bending moment is zero at a point of inflection, the freebody diagram would indicate that the middle half of the fixed ended is equivalent to a hinged column having an effective length $L_e = L / 2$.

The four different cases which we have considered so far are:

- (a) Both ends pinned
- (b) Both ends fixed
- (c) One end fixed, other free
- (d) One end fixed and other pinned



Source: <http://nptel.ac.in/courses/Webcourse-contents/IIT-Roorkee/strength%20of%20materials/homepage.htm>