Design principles for thick cylinders

Instructional Objectives:

At the end of this lesson, the students should have the knowledge of:

- Failure theories applied to thick walled pressure vessels.
- Variation of wall thickness with internal pressure based on different failure theories.
- Failure criterion of prestressed thick cylinders.
- Comparison of wall thickness variation with internal pressure for solid wall, single jacket and laminated thick walled cylinders.
- Failure criterion for thick walled cylinders with autofrettage.

9.3.1 Application of theories of failure for thick walled pressure vessels.

Having discussed the stresses in thick walled cylinders it is important to consider their failure criterion. The five failure theories will be considered in this regard and the variation of wall thickness to internal radius ratio t/r_i or radius ratio r_o/r_i with p/σ_{yp} for different failure theories would be discussed. A number of cases such as p_o =0, p_i =0 or both non-zero p_o and p_i are possible but here only the cylinders with closed ends and subjected to an internal pressure only will be considered, for an example.

9.3.1.1 Maximum Principal Stress theory

According to this theory failure occurs when maximum principal stress exceeds the stress at the tensile yield point. The failure envelope according to this failure mode is shown in **figure-9.3.1.1.1** and the failure criteria are given by $\sigma_1 = \sigma_2 = \pm\, \sigma_{yp}$. If p_o =0 the maximum values of circumferential and radial stresses are given by

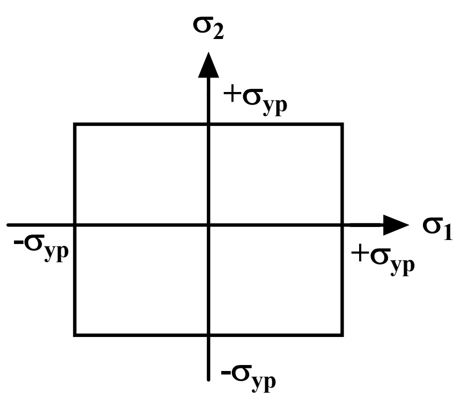
$$\sigma_{\theta(\max)}\Big|_{r=r_i} = p_i \frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} \qquad \sigma_{r(\max)}\Big|_{r=r_i} = -p_i \tag{1}$$

Here both σ_θ and σ_r are the principal stresses and σ_θ is larger. Thus the condition for failure is based on σ_θ and we have

$p_i \frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} = \sigma_{yp}$ where σ_{yp} is the yield stress.

This gives $\frac{t}{r_i} = \sqrt{\frac{1+ p_i/\sigma_{yp}}{1- p_i/\sigma_{yp}}} - 1$

(2)



9.3.1.1.1F- Failure envelope according to Maximum Principal Stress Theory.

9.3.1.2 Maximum Shear Stress theory

According to this theory failure occurs when maximum shear stress exceeds the maximum shear stress at the tensile yield point. The failure envelope according to this criterion is shown in **figure- 9.3.1.2.1** and the maximum shear stress is given by

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2}$$

where the principal stresses σ_1 and σ_2 are given by

$$\sigma_1 = \sigma_\theta = p_i \frac{r_o^2 + r_i^2}{r_o^2 - r_i^2}$$

$$\sigma_2 = \sigma_r = -p_i$$

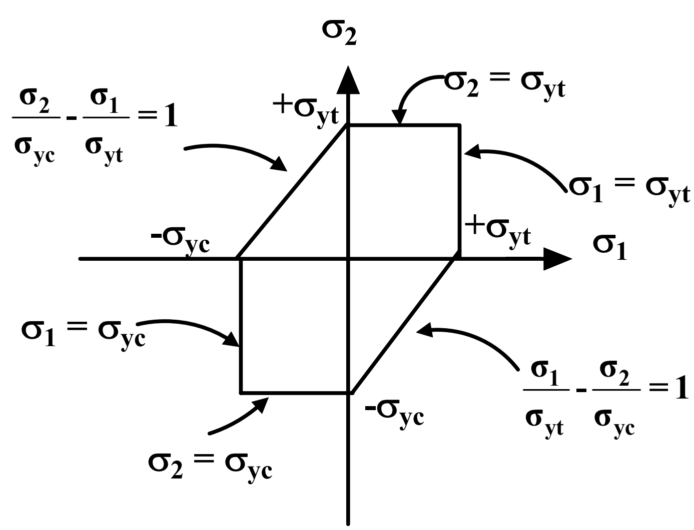
Here σ_1 is tensile and σ_2 is compressive in nature. τ_{max} may therefore be given by

$$\tau_{max} = p_i \frac{r_o^2}{r_o^2 - r_i^2}$$

(3)

and since the failure criterion is $\tau_{max} = \sigma_{yp} / 2$ we may write

$$\frac{t}{r_i} = \sqrt{\frac{1}{1 - 2\left(p_i / \sigma_{yp}\right)}} - 1 \qquad (4)$$



9.3.1.2.1F- Failure envelope according to Maximum Shear Stress theory.

9.3.1.3 <u>Maximum Principal Strain theory</u>

According to this theory failure occurs when the maximum principal strain exceeds the strain at the tensile yield point.

$$\varepsilon_1 = \frac{1}{E}\left\{\sigma_1 - \nu\left(\sigma_2 + \sigma_3\right)\right\} = \varepsilon_{yp} \text{ and this gives } \sigma_1 - \nu\left(\sigma_2 + \sigma_3\right) = \sigma_{yp}$$

where ε_{yp} and σ_{yp} are the yield strain and stress respectively. Following this the failure envelope is as shown in **figure-9.3.1.3.1.** Here the three principle stresses can be given as follows according to the standard 3D solutions:

$$\sigma_1 = \sigma_\theta = p_i \frac{r_o^2 + r_i^2}{r_o^2 - r_i^2}, \ \sigma_2 = \sigma_r = -p_i \text{ and } \sigma_3 = \sigma_z = \frac{p_i r_i^2}{r_o^2 - r_i^2}$$

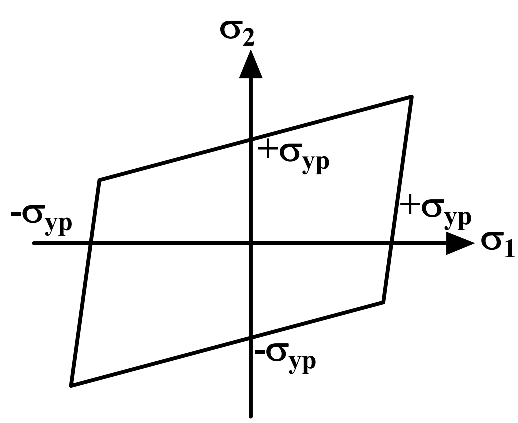
(5)

The failure criterion may now be written as

$$p_i\left(\frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} + \nu - \frac{\nu r_i^2}{r_o^2 - r_i^2}\right) = \sigma_{yp} \text{ and this gives}$$

$$\frac{t}{r_i} = \sqrt{\frac{1+\left(1-2\nu\right)p_i/\sigma_{yp}}{1-\left(1+\nu\right)p_i/\sigma_{yp}}} - 1$$

(6)



9.3.1.3.1F- Failure envelope according to Maximum Principal Strain theory

9.3.1.4 Maximum Distortion Energy Theory

According to this theory if the maximum distortion energy exceeds the distortion energy at the tensile yield point failure occurs. The failure envelope is shown in **figure-9.3.1.4.1** and the distortion energy E_d is given by

$$E_d = \frac{1+\nu}{6E}\left\{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right\}$$

Since at the uniaxial tensile yield point $\sigma_2 = \sigma_3 = 0$ and $\sigma_1 = \sigma_{yp}$

E_d at the tensile yield point $= \frac{1+\nu}{3E}\sigma_{yp}^2$

We consider $\sigma_1 = \sigma_\theta$, $\sigma_2 = \sigma_r$ and $\sigma_3 = \sigma_z$ and therefore

$$\sigma_1 = p_i\frac{r_o^2+r_i^2}{r_o^2-r_i^2}$$

$$\sigma_r = -p_i \qquad \sigma_z = \frac{p_i r_i^2}{r_o^2-r_i^2}$$

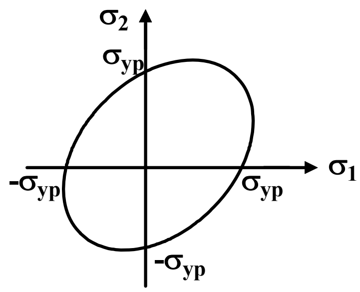
(7)

The failure criterion therefore reduces to

$$\frac{p_i}{\sigma_{yp}} = \frac{1}{\sqrt{3}}\left(\frac{r_o^2-r_i^2}{r_o^2}\right)$$ which gives

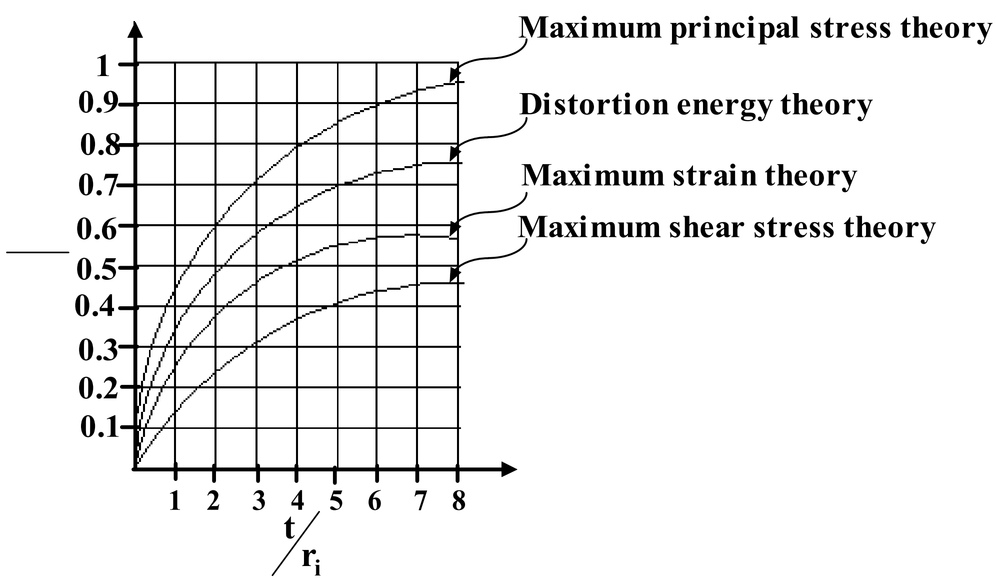
$$\frac{t}{r_i} = \sqrt{\frac{1}{1-\sqrt{3}\,p_i/\sigma_{yp}}}-1$$

(8)



9.3.1.4.1F- Failure envelope according to Maximum Distortion Energy Theory

Plots of p_i/σ_{yp} and t/r_i for different failure criteria are shown in **figure-9.3.1.4.2**.



9.3.1.4.2F- Comparison of variation of —— against t/r_i for different failure criterion.

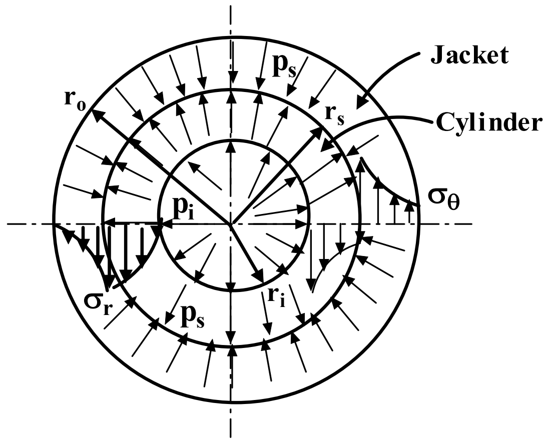
The criteria developed and the plots apply to thick walled cylinders with internal pressure only but similar criteria for cylinders with external pressure only or in case where both internal and external pressures exist may be developed. However, on the basis of these results we note that the rate of increase in p_i/σ_{yp} is small at large values of t/r_i for all the failure modes considered. This means that at higher values of p_i small increase in pressure requires large increase in wall thickness. But since the stresses near the outer radius are small, material at the outer radius for very thick wall cylinders are ineffectively used. It is therefore necessary to select materials so that p_i/σ_{yp} is reasonably small. When this is not possible prestressed cylinders may be used.

All the above theories of failure are based on the prediction of the beginning of inelastic deformation and these are strictly applicable for ductile materials under static loading. Maximum principal stress theory is widely used for brittle materials which normally fail by brittle fracture.

In some applications of thick cylinders such as, gun barrels no inelastic deformation can be permitted for proper functioning and there design based on maximum shear stress theory or maximum distortion energy theory are acceptable. For some pressure vessels a satisfactory function is maintained until inelastic deformation that starts from the inner radius and spreads completely through the wall of the cylinder. Under such circumstances none of the failure theories would work satisfactorily and the procedure discussed in section lesson 9.2 is to be used.

9.3.1.5 Failure criteria of pre-stressed thick cylinders

Failure criteria based on the three methods of pre-stressing would now be discussed. The radial and circumferential stresses developed during shrinking a hollow cylinder over the main cylinder are shown in **figure-9.3.1.5.1**.



9.3.1.5.1F- Distribution of radial and circumferential stresses in a composite thick walled cylinder subjected to an internal pressure.

Following the analysis in section 9.2 the maximum initial (residual) circumferential stress at the inner radius of the cylinder due to the contact pressure p_s is

$$\sigma_{\theta}\Big|_{r=r_i} = -2p_s \frac{r_s^2}{r_o^2 - r_s^2}$$

and the maximum initial (residual) circumferential stress at the inner radius of the jacket due to contact pressure p_s is

$$\sigma_{\theta}\Big|_{r=r_s} = p_s \frac{r_o^2 + r_s^2}{r_o^2 - r_s^2}$$

Superposing the circumferential stresses due to p_i (considering the composite cylinder as one) the total circumferential stresses at the inner radius of the cylinder and inner radius of the jacket are respectively

$$\sigma_{\theta}\Big|_{r=r_i} = -2p_s \frac{r_s^2}{r_s^2 - r_i^2} + p_i \frac{r_o^2 + r_i^2}{r_o^2 - r_i^2}$$

$$\sigma_{\theta}\Big|_{r=r_s} = p_s \frac{r_o^2 + r_s^2}{r_o^2 - r_s^2} + p_i \frac{r_i^2}{r_s^2}\left(\frac{r_o^2 + r_s^2}{r_o^2 - r_i^2}\right)$$

These maximum stresses should not exceed the yield stress and therefore we may write

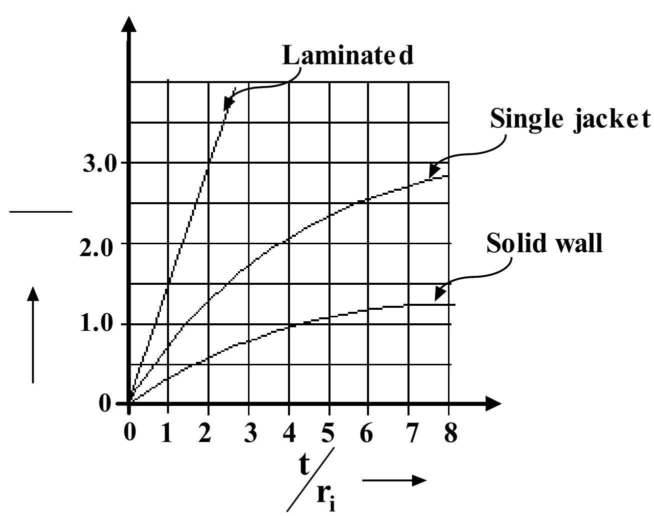
$$-2p_s \frac{r_s^2}{r_s^2 - r_i^2} + p_i \frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} = \sigma_{yp} \qquad (9)$$

$$p_s \frac{r_o^2 + r_s^2}{r_o^2 - r_s^2} + p_i \frac{r_i^2}{r_s^2}\left(\frac{r_o^2 + r_s^2}{r_o^2 - r_i^2}\right) = \sigma_{yp} \qquad (10)$$

It was shown in section-9.2 that the contact pressure p_s is given by

$$p_s = \frac{E\delta}{r_s\left[\frac{r_o^2 + r_s^2}{r_o^2 - r_s^2} + \frac{r_s^2 + r_i^2}{r_s^2 - r_i^2}\right]} \qquad (11)$$

From (9), (10) and (11) it is possible to eliminate p_s and express t/r_i in terms of p_i/σ_{yp} and this is shown graphically in **figure-9.3.1.5.2**.



9.3.1.5.2F- Plot of p_i/σ_{yp} vs t/r_i for laminated multilayered, single jacket and solid wall cylinders.

This shows that even with a single jacket there is a considerable reduction in wall thickness and thus it contributes to an economic design.

As discussed earlier autofrettage causes yielding to start at the inner bore and with the increase in pressure it spreads outwards. If now the pressure is released the outer elastic layer exerts radial compressive pressure on the inner portion and this in turn causes radial compressive stress near the inner portion and tensile stress at the outer portion. For a given fluid pressure during autofrettage a given amount of inelastic deformation is produced and therefore in service the same fluid pressure may be used without causing any additional elastic deformation.

The self hooping effect reaches its maximum value when yielding just begins to spread to the outer wall. Under this condition the cylinder is said to have reached a fully plastic condition and the corresponding internal fluid pressure is known as fully plastic pressure, say, p_f . This pressure may be found by using the reduced equilibrium equation (3) in section- 9.2.1 which is reproduced here for convenience

$$\sigma_\theta = \sigma_r + r\frac{d\sigma_r}{dr}$$

(12)

Another equation may be obtained by considering that when the maximum shear stress at a point on the cylinder wall reaches shear yield value τ_{yp} it remains constant even after further yielding. This is given by

$$\frac{1}{2}\left(\sigma_\theta - \sigma_r\right) = \tau_{yp}$$

(13)

However experiments show that fully plastic pressure is reached before inelastic deformation has spread to every point on the wall. In fact Luder's lines appear first. Luder's lines are spiral bands across the cylinder wall such that the material between the bands retains elasticity. If the cylinder is kept under fully plastic pressure for several hours uniform yielding across the cylinder wall would occur.

This gives $\frac{d\sigma_r}{dr} = 2\tau_{yp} / r$ and on integration we have

$$\sigma_r = 2\tau_{yp} \log r + c$$

Applying the boundary condition at r = r_o σ_r = 0 we have

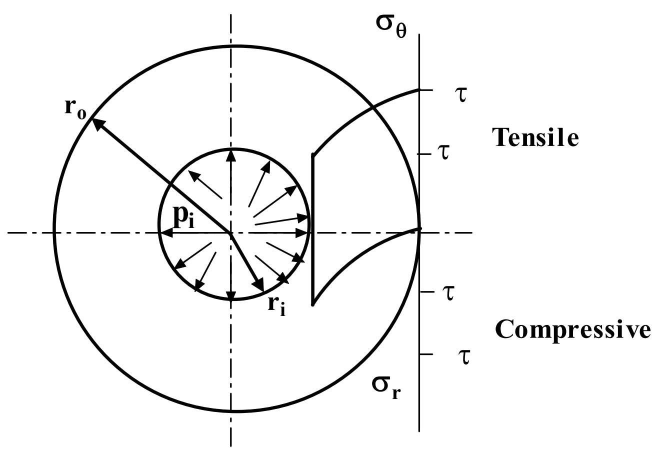
$$\sigma_r = 2\tau_{yp} \log\left(\frac{r}{r_o}\right) \text{ and } \sigma_\theta = 2\tau_{yp}\left\{1 + \log\left(\frac{r}{r_o}\right)\right\}$$

(14)

Also applying the boundary condition at r = r_i σ_r = - p_f we have

$$p_f = -2\tau_{yp} \log\left(\frac{r_i}{r_o}\right) \qquad (15)$$

Since the basic equations are independent of whether the cylinders are open or closed ends, the expressions for σ_r and σ_θ apply to both the conditions. The stress distributions are shown in **figure- 9.3.1.5.3**.



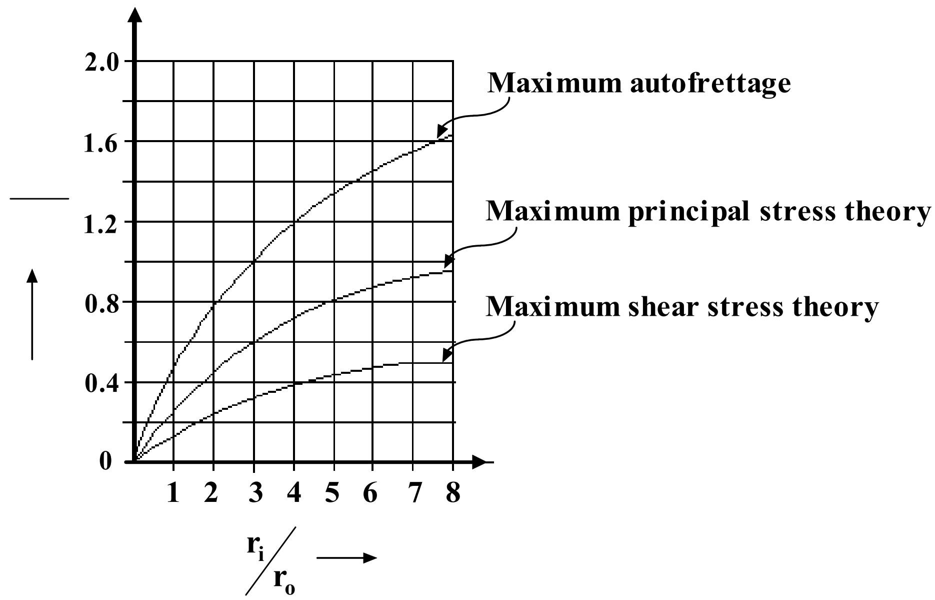
9.3.1.5.3F- Stress distribution in a thick walled cylinder with autofrettage

If we roughly assume that $2\tau_{yp} = \sigma_{yp}$ we have

$$\frac{p_f}{\sigma_{yp}} = -\log\left(\frac{r_i}{r_o}\right)$$

(16)

The results of maximum principal stress theory and maximum shear stress theory along with the fully plastic results are replotted in **figure 9.3.1.5.4** where we may compare the relative merits of different failure criteria. It can be seen that cylinders with autofrettage may endure large internal pressure at relatively low wall thickness.



9.3.1.5.4F- Plots of p_i/σ_{yp} vs r_i/r_o for maximum shear stress theory, maximum principal stress theory and maximum autofrettage.

Finally it must be remembered that for true pressure vessel design it is essential to consult Boiler Codes for more complete information and guidelines. Pressure vessels can be extremely dangerous even at relatively low pressure and therefore the methodology stated here is a rough guide and should not be considered to be a complete design methodology.

9.3.2Problems with Answers

Q.1: Determine the necessary thickness of the shell plates of 2.5m diameter boiler with the internal pressure of 1MPa. The material is mild steel with a tensile strength of 500MPa. Assuming an efficiency of the longitudinal welded joint to be 75% and a factor of safety of 5 find the stress in the perforated steel plate.

A.1:

Considering that the boiler design is based on thin cylinder principles the shell thickness is given by

$t = \frac{pr}{\sigma_{ty}\eta}$ where r is the boiler radius and η is the joint efficiency.

This gives

$$t = \frac{10^6 x1.25}{\left(\frac{500}{5}\right)x10^6 x0.75} = 0.0166\text{m} = 16.6\,\text{mm, say } 20\text{mm.}$$

The stress in the perforated plate is therefore given by $\sigma = \frac{pr}{t}$ i.e. 62.5MPa

Q.2: A hydraulic cylinder with an internal diameter 250mm is subjected to an internal pressure of 10 MPa. Determine the wall thickness based on (a) Maximum principal stress theory, b) Maximum shear stress theory and c)

Maximum distortion energy theory of failure. Compare the results with wall thickness calculated based on thin cylinder assumption. Assume the yield stress of the cylinder material to be 60 MPa.

A.2:

Considering that the hydraulic cylinders are normally designed on the thick cylinder assumption we have from section 9.3.1.1 for Maximum Principal stress Theory we have

$$t = r_i\left(\sqrt{\frac{1+\frac{p_i}{\sigma_{yp}}}{1-\frac{p_i}{\sigma_{yp}}}}-1\right)$$

Here $\frac{p_i}{\sigma_{yp}} = 10/60 \simeq 0.167$ and r_i = 125 mm. This gives t = 22.9mm, say 23 mm

From section 9.3.1.2 for Maximum Shear Stress theory we have

$$t = r_i\left(\sqrt{\frac{1}{1-2\left(\frac{p_i}{\sigma_{yp}}\right)}}-1\right)$$

With $\frac{p_i}{\sigma_{yp}} \approx 0.167$ and r_i = 125 mm, t = 28.2 mm, say 29 mm.

From section 9.3.1.4 for maximum distortion energy theory we have

$$t = r_i\left(\frac{1}{\sqrt{1-\sqrt{3}\left(\frac{p_i}{\sigma_{yp}}\right)}}\ -1\right)$$

with $\frac{p_i}{\sigma_{yp}} \approx 0.167$ and r_i = 125mm t = 23.3 mm, say 24 mm.

Considering a thin cylinder t = $r_i \left(\frac{p_i}{\sigma_{yp}}\right)$ and this gives t = 20.875mm, say 21 mm.

The thin cylinder approach yields the lowest wall thickness and this is probably not safe. The largest wall thickness of 29mm predicted using the maximum shear stress theory is therefore adopted.

Q.3: A cylinder with external diameter 300mm and internal diameter 200mm is subjected to an internal pressure of 25 MPa. Compare the relative merits of a single thick walled cylinder and a composite cylinder with the inner cylinder whose internal and external diameters are 200mm and 250 mm respectively. A tube of 250 mm internal diameter and 300mm external diameter is shrunk on the main cylinder. The safe tensile yield stress of the material is 110 MPa and the stress set up at the junction due to shrinkage should not exceed 10 MPa.

A.3:

We first consider the stresses set up in a single cylinder and then in a composite cylinder.

Single cylinder

The boundary conditions are

at r = 150mm $\sigma_r = 0$ and at r = 100mm σ_r = - 20MPa

Using equation (10) in section 9.2.1

$C_1 + \frac{C_2}{0.0225} = 0$ and $C_1 + \frac{C_2}{0.01} = -20$

This gives C_1= 16 and C_2 = -0.36

The hoop stress at r = 100mm and r = 150 mm are **52 MPa** and **32 MPa** respectively.

Stress in the composite cylinder

The stresses in the cylinder due to shrinkage only can be found using the following boundary conditions

at r = 150mm $\sigma_r = 0$ and at r = 125mm $\sigma_r = -10$MPa

Following the above procedure the hoop stress at r = 150 mm and r = 125mm are **45.7MPa** and **55.75MPa** respectively.

The stress in the inner cylinder due to shrinkage only can be found using the following boundary conditions

at r = 100mm $\sigma_r = 0$ and at r = 125mm $\sigma_r = -10$MPa

This gives the hoop stress at r = 100mm and r = 125mm to be **- 55.55 MPa** and **– 45.55 MPa** respectively.

Considering the internal pressure only on the complete cylinder the boundary conditions are

at r = 150mm $\sigma_r = 0$ and at r = 100mm $\sigma_r = -25$ MPa

This gives

$(\sigma_\theta)_{r=150mm}$ = **40MPa** $(\sigma_\theta)_{r=125mm}$ = **49 MPa** $(\sigma_\theta)_{r=100mm}$ = **65MPa.**

Resultant stress due to both shrinkage and internal pressure

Outer cylinder

$(\sigma_\theta)_{r=150mm}$ = 40 +45.7 = **85.7 MPa**

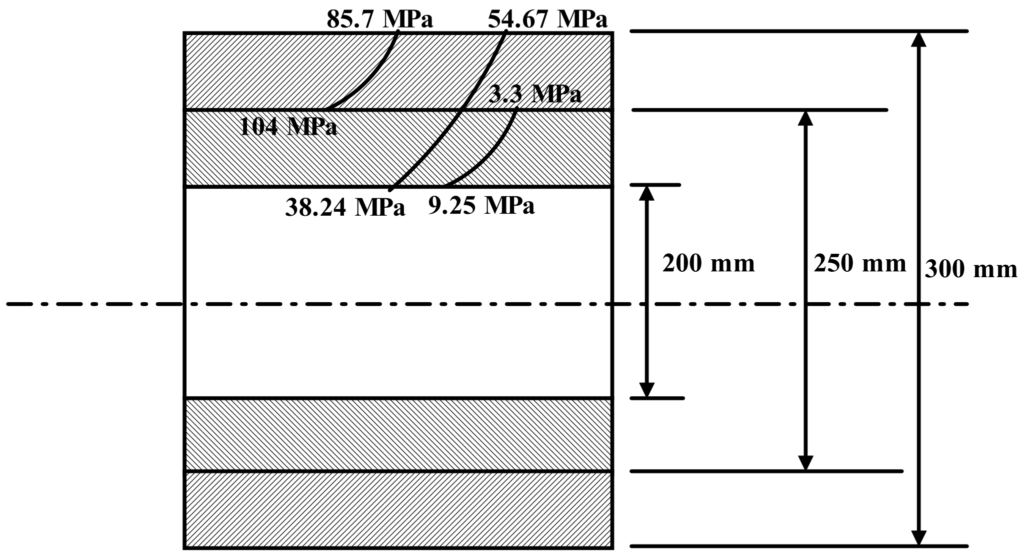
$(\sigma_\theta)_{r=125mm}$ = 49+55.75 = **104.75 MPa**

Inner cylinder

$(\sigma_\theta)_{r=125mm}$ = 49 -45.7 = **3.3 MPa**

$(\sigma_\theta)_{r=100mm}$ = 65 - 55.75 = **9.25 MPa**

The stresses in both the single cylinder and the composite are within the safe tensile strength of the material. However in the single cylinder the stress gradient is large across the wall thickness whereas in the composite cylinder the stress variation is gentle. These results are illustrated in **figure- 9.3.2.1**



9.3.2.1F- Stress gradients (circumferential) in the inner and outer cylinders as well as the gradient across the wall of a single cylinder.

9.3.3 Summary of this Lesson

The lesson initially discusses the application of different failure theories in thick walled pressure vessels. Failure criterion in terms of the ratio of wall thickness to the internal radius and the ratio of internal pressure to yield stress have been derived for different failure criterion. Failure criterion for prestressed composite cylinders and cylinders with autofrettage have also been derived. Finally comparisons of different failure criterion have been discussed.

9.3.4References for Module-9

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