Derivation of the Navier-Stokes Equation

- There are three kinds of forces important to fluid mechanics: gravity (body force), pressure forces, and viscous forces (due to friction). Gravity force,
- **Body forces** act on the entire element, rather than merely at its surfaces. The only body force to be considered here is that due to gravity. By convention, gravity acts in the negative z-direction, i.e. *downward*.
- **Pressure forces** act inward and normal to the surfaces of the element, and have been discussed previously.
- Finally, there are **viscous forces**, due to friction acting on the fluid element because of viscosity in the fluid. These viscous forces are surface forces, like the forces due to pressure, but can act in any direction on the surface. In other words, viscous forces at a surface can have both normal and tangential (or *shear*) components.

Viscous forces for incompressible Newtonian fluids

- Here, consider only ***Newtonian*** fluids. A Newtonian fluid is one where the stress is linearly proportional to the strain. Most common fluids are Newtonian, such as water, air, gasoline, oil, etc. However, some fluids have a nonlinear relationship between stress and strain. These fluids are called ***non-Newtonian***. Some examples of non-Newtonian fluids are cake batter and bread dough. Blood also has some non-Newtonian properties. It turns out that the net viscous force per unit volume for an incompressible Newtonian fluid is

$$\vec{f}_{visc} = \mu \nabla^2 \vec{V}$$

where the right hand side is the laplacian of the velocity vector. Note that in hydrostatics, where the velocity is identically zero, there is no viscous force, regardless of the value of viscosity.

- The sum of all the forces on the element must equal the mass of the element times its acceleration (Newton's second law). On a per unit volume basis, the equation of fluid motion is then

$$\sum \vec{f} = \rho\vec{a} = \vec{f}_{grav} + \vec{f}_{press} + \vec{f}_{visc}$$
$$\rho\vec{a} = \rho\vec{g} - \vec{\nabla}p + \mu\nabla^2\vec{V}$$

  The above equation is the famous ***Navier-Stokes equation***, valid for incompressible Newtonian flows. Normally, the acceleration term on the left is expanded as the material acceleration when writing this equation, i.e.

$$\rho\left(\frac{\partial\vec{V}}{\partial t} + u\frac{\partial\vec{V}}{\partial x} + v\frac{\partial\vec{V}}{\partial y} + w\frac{\partial\vec{V}}{\partial z}\right) = \rho\vec{g} - \vec{\nabla}p + \mu\nabla^2\vec{V}$$

- Solutions of the full Navier-Stokes equation will be discussed in a later module. For now, consider some simplifications of this equation.
  - ***Hydrostatics*** (fluid statics) - This is the simplest possible case, namely when the fluid is either completely at rest or moving at a constant speed with no acceleration. In such as case, both acceleration and any velocity derivatives are zero, and the Navier-Stokes equation reduces to

$$\vec{\nabla} p = \rho \vec{g}$$

- ***Rigid body motion*** **-** The next simplest case is when there is an acceleration of the fluid, but the entire fluid mass moves together rigidly as one big chunk. Examples include rigid body acceleration of a container of fluid in a straight line and rigid body rotation of a fluid about some axis. In such cases, although there is viscosity in the fluid, it is not felt since no fluid particles ever "rub against each other." In other words, there can be no viscous shear in rigid body motion, and the Navier-Stokes equation reduces to

$$\vec{\nabla} p = \rho\left(\vec{g} - \vec{a}\right)$$

- ***Inviscid fluid flow*** - In some practical applications, the effects of viscosity are negligible. The viscous terms in the Navier Stokes equation are then neglected, and the equation reduces to the ***Euler equation***,

$$\rho \vec{a} = -\vec{\nabla} p + \rho \vec{g}$$

Source:

http://www.mne.psu.edu/cimbala/Learning/Fluid/Navier_Stokes/navier_stokes_derivation.htm