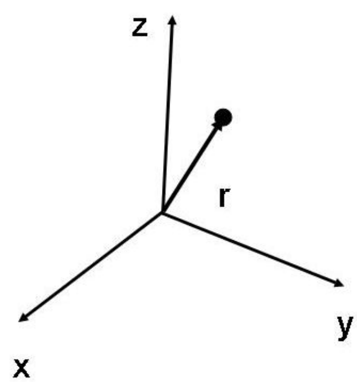
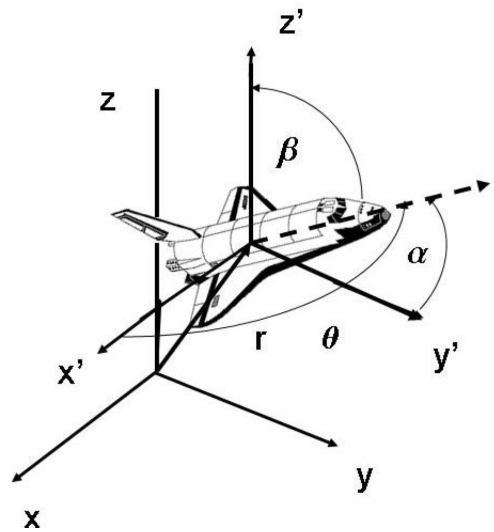
Degrees of Freedom and Constraints, Rectilinear Motion

Degrees of Freedom

Degrees of freedom refers to the number of independent spatial coordinates that must be specified to determine the position of a body. If the body is a point mass, only three coordinates are required to determine its position. On the other hand, if the body is extended, such as an aircraft, three position coordinates and three angular coordinates are required to completely specify its position and orientation in space.

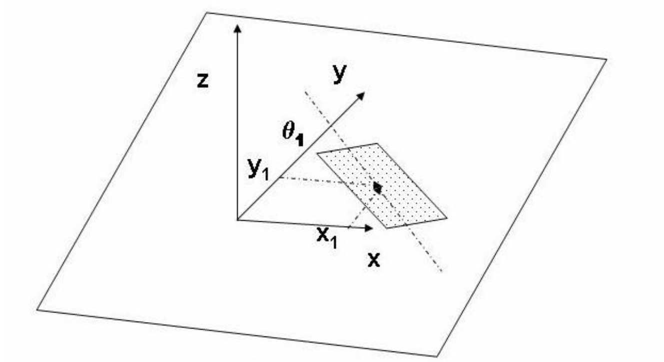




Kinematic Constraints

In many situations the number of independent coordinates will be reduced below this number, either because the number of spacial dimensions is reduced or because there are relationships specified among the spatial coordinates. When setting up problems for solution it is useful to think of these relationships as constraints.

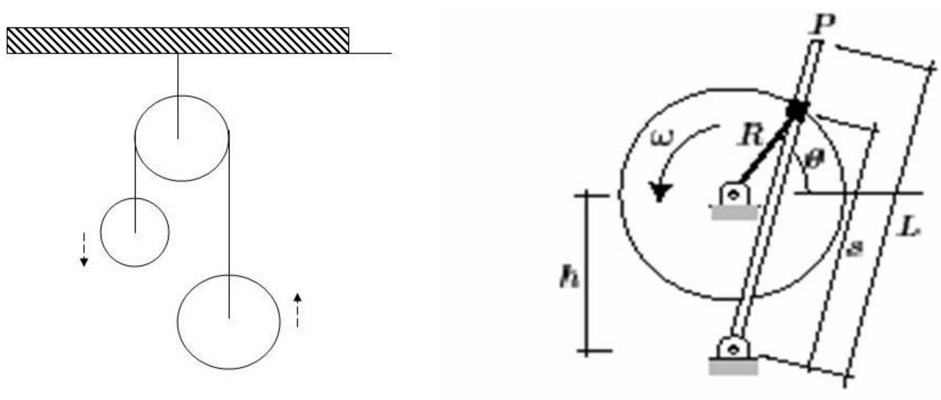
For example, if a point mass is constrained to move in a plane (two dimensions) the number of spatial coordinates necessary to describe its motion is two. If instead of being a point mass, this body has extended dimensions, such as a flat plate confined to a plane, it requires three coordinates to specify its position and orientation: two position coordinates and one angular coordinate.



x, y plane

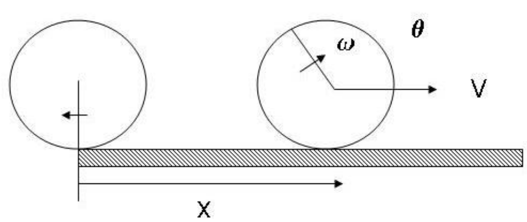
If a particle is confined to move on a curve in either two or three dimensions, such as a bead moving on a wire, the number of independent coordinates necessary to describe its motion is one.

Another source of constraints on the motion of particles is connections between them. For example, the two particle connected by a cable passing over a pulley are constrained to move in equal and opposite directions. More complex arrangements are possible and can be analyzed using these ideas. Two gears in contact are constrain to move together according to their individual geometry.



SINGLE DEGREE OF FREEDOM SYSTEMS

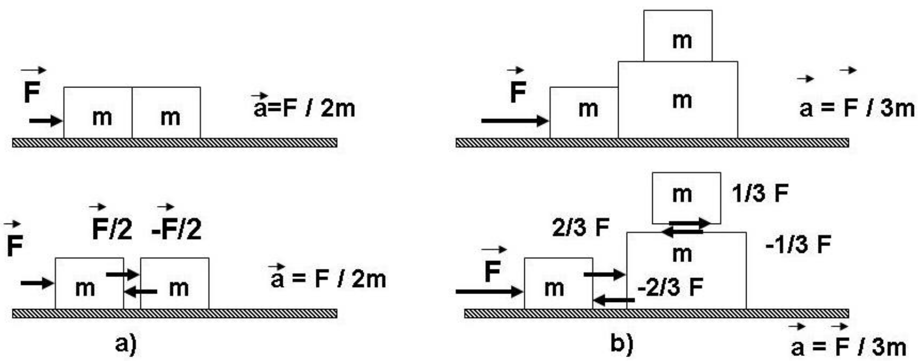
A cylinder rolling on a plane is constrained in two ways. Contact with the plane reduces the two-dimensional motion to one spatial coordinate along the plane, and the constraint of rolling provides a relationship between the angular coordinates and the spatial position, resulting in a single degree of freedom system.



Internal Force-Balance Constraints

Another type of constraint occurs when we consider the of a system of particles and the necessary force balance that occurs between the parts. These constraints follow directly from Newton's third law: the force of action and reaction between two bodies are equal in magnitude and opposite in direction. We will pursue these ideas in greater depth later in the course. For now, we will give a simple example to illustrate the principle.

Consider the systems shown in **a)** and **b)**.



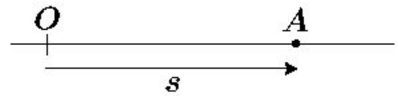
Internal Force Balance Constraints

System **a)** consists of two masses **m** in contact resting on a frictionless plane in the presence of gravity. A force **F** is applied to mass 1, and it is obvious that the two masses will accelerate at $a = \mathbf{F}/(2m)$. If we look at the two masses separately, we can determine what internal force must exist between them to cause the motion. It is clear that each mass feels a net force of $\mathbf{F/2}$, since its acceleration is $a = \mathbf{F}/(2m)$. This net force arises because between the two masses there is an equal and opposite force $\mathbf{F/2}$ acting across the interface. Another way to look at this is that the interface between the bodies is a "body" of zero mass, and therefore can have no net force acting upon it otherwise its acceleration would be infinite.

System **b)** is a bit more complex, primarily because the forces between one mass and the mass above it are shear forces and must be supplied by friction. Assuming that the friction coefficient is large enough to accelerate the three masses an equal amount given by $a = F/(3m)$, by the reasoning we have discussed, the force balance is as sketched in **b)**: equal and opposite normal forces $\mathbf{F\ 2/3}$ on the vertical surfaces, and equal and opposite shear forces $\mathbf{F\ 1/3}$ on the horizontal surfaces.

Rectilinear Motion

In many case we can get an exact expression for the position of a particle as a function of time. We start by considering the simple motion of a particle along a straight line. The position of particle A at any instant can be specified by the coordinate s with origin at some fixed point O.



The instantaneous velocity is

$$v = \frac{ds}{dt} = \dot{v} \ . \tag{1}$$

We will be using the "dot" notation, to indicate time derivative, e.g. $(\dot{}) \equiv d/dt$. Here, a positive v means that the particle is moving in the direction of increasing s, whereas a negative v, indicates that the particle is moving in the opposite direction. The acceleration is

$$a = \frac{dv}{dt} = \dot{v} = \frac{d^2 s}{dt^2} = \ddot{s} \ . \tag{2}$$

The above expression allows us to calculate the speed and the acceleration if s and/or v are given as a function of t, i.e. $s(t)$ and $v(t)$. In most cases however, we will know the acceleration and then, the velocity and the position will have to be determined from the above expressions by integration.

Determining the velocity from the acceleration

From $a(t)$

If the acceleration is given as a function of t, $a(t)$, then the velocity can be determined by simple integration of equation (2),

$$v(t) = v_0 + \int_{t_0}^{t} a(t)\, dt \ . \tag{3}$$

Here, v_0 is the velocity at time t_0, which is determined by the initial conditions.

From $a(v)$

If the acceleration is given as a function of velocity $a(v)$, then, we can still use equation (2), but in this case we will solve for the time as a function of velocity,

$$t(v) = t_0 + \int_{v_0}^{v} \frac{dv}{a(v)} \ . \tag{4}$$

Once the relationship $t(v)$ has been obtained, we can, in principle, solve for the velocity to obtain $v(t)$. A typical example in which the acceleration is known as a function of velocity is when aerodynamic drag forces are present. Drag forces cause an acceleration which opposes the motion and is typically of the form $a(v) \propto v^2$ (the sign "$\propto$" means *proportional to*, that is, $a(v) = \kappa v^2$ for some κ, which is not a function of velocity).

From $a(s)$

When the acceleration is given as a function of s then, we need to use a combination of equations (1) and (2), to solve the problem. From

$$a = \frac{dv}{dt} = \frac{dv}{ds} * \frac{ds}{dt} = v\frac{dv}{ds} \tag{5}$$

we can write

$$a\,ds = v\,dv \ . \tag{6}$$

This equation can now be used to determine v as a function of s,

$$v^2(s) = v_0^2 + 2\int_{s_0}^{s} a(s)\,ds \ . \tag{7}$$

where, v_0, is the velocity of the particle at point s_0. Here, we have used the fact that,

$$\int_{v_0}^{v} v\,dv = \int_{v_0}^{v} d(\frac{v^2}{2}) = \frac{v^2}{2} - \frac{v_0^2}{2} \ .$$

A classical example of an acceleration dependent on the spatial coordinate s, is that induced by a deformed linear spring. In this case, the acceleration is of the form $a(s) \propto s$.

Of course, when the acceleration is constant, any of the above expressions (3, 4, 7), can be employed. In this case we obtain,

$$v = v_0 + a(t - t_0), \quad \text{or} \quad v^2 = v_0^2 + 2a(s - s_0) \ .$$

If $a = g$, this reduces to the familiar

$$v = v_0 + g(t - t_0), \quad \text{or} \quad v^2 = v_0^2 + 2g(s - s_0) \ .$$

Determining the position from the velocity

Once we know the velocity, the position can be found by integrating $ds = vdt$ from equation (1). Thus, when the velocity is known as a function of time we have,

$$s = s_0 + \int_{t_0}^{t} v(t)\,dt \ . \tag{8}$$

If the velocity is known as a function of position, then

$$t = t_0 + \int_{s_0}^{s} \frac{ds}{v(s)} \ . \tag{9}$$

Here, s_0 is the position at time t_0.

It is worth pointing out that equation (6), can also be used to derive an expression for $v(s)$, given $a(v)$,

$$s - s_0 = \int_{s_0}^{s} ds = \int_{v_0}^{v} \frac{v}{a(v)}\,dv \ . \tag{10}$$

This equation can be used whenever equation (4) is applicable and gives $v(s)$ instead of $t(v)$. For the case of constant acceleration, either of equations (8, 9), can be used to obtain,

$$s = s_0 + v_0(t - t_0) + \frac{1}{2}a(t - t_0)^2 \ .$$

In many practical situations, it may not be possible to carry out the above integrations analytically in which case, numerical integration is required. Usually, numerical integration will also be required when either the velocity or the acceleration depend on more than one variable, i.e. $v(s,t)$, or, $a(s,v)$.

Example ***Reentry, Ballistic Coefficient, Terminal Velocity***

Terminal Velocity

Terminal velocity occurs when the acceleration becomes zero and the velocity Consider an air-dropped payload starting from rest. The force on the body is a combination of gravity and air drag and has the form

$$F = mg - \frac{1}{2}\rho v^2 * C_D * A \tag{11}$$

Applying Newton's law and solving for the acceleration a we obtain

$$a = g - \frac{1}{2}\rho v^2 * \frac{C_D * A}{m} \tag{12}$$

The quantity $\frac{m}{C_D * A}$ characterizes the combined effect of body shape and mass on the acceleration; it is an important parameter in the study of reentry; it is called the Ballistic Coefficient. It is defined as $\beta = \frac{m}{C_D * A}$. Unlike many coefficients that appear in aerospace problems, the Ballistic Coefficient is not non-dimensional, but has units of $mass/length^2$ or $kg/meters^2$ in mks units. Also, in some applications, the ballistic coefficient is defined as the inverse, $B = \frac{C_D * A}{m}$, so it pays to be careful in its application. Equation 13 then becomes

$$a = g - \frac{1}{2}\rho v^2 * /\beta \tag{13}$$

Terminal velocity occurs when the force of gravity equals the drag on the object resulting in zero acceleration. This balance gives the terminal velocity as

$$v_{terminal} = \sqrt{\frac{2gm}{C_D A \rho}} = \sqrt{\frac{2g\beta}{\rho}} \tag{14}$$

For the Earth, atmospheric density at sea level is $\rho = 1.225 kg/m^3$; we shall deal with the variation of atmospheric density with altitude when we consider atmospheric reentry of space vehicles. Typical value of β range from $\beta = 1$ (Assuming $C_D = .5$, a tennis ball has $\beta = 35$.) to $\beta = 1000$ for a reentry vehicle. As an example, consider a typical case of $\beta = 225$, where the various parameters then give the following expression for the acceleration

$$a = g - 0.002725 v^2 \text{ m/s} .$$

Here $g = 9.81\text{m/s}^2$, is the acceleration due to gravity and v is the downward velocity. It is clear from this expression that initially the acceleration will be g. Therefore, the velocity will start to increase and keep on increasing until $a = 0$, at which point the velocity will stay constant. The terminal velocity is then given by,

$$0 = g - 0.002725\, v_f^2 \quad \text{or} \qquad v_f = 60\text{m/s} .$$

To determine the velocity as a function of time, the acceleration can be re-written introducing the terminal velocity as, $a = g(1 - (v/v_f)^2)$. We then use expression (4), and write

$$t = \frac{1}{g}\int_0^v \frac{dv}{1-(v/v_f)^2} = \frac{1}{g}\int_0^v (\frac{1/2}{1+(v/v_f)} + \frac{1/2}{1-(v/v_f)})d\,v = \frac{v_f}{2g}\ln\left(\frac{v_f+v}{v_f-v}\right) \quad .$$

Solving for v we obtain,

$$v = v_f\,\frac{e^{2gt/v_f}-1}{e^{2gt/v_f}+1}\ \text{m/s} \ .$$

We can easily verify that for large t, $v = v_f$. We can also find out how long does it take for the payload to reach, say, 95% of the terminal velocity,

$$t = \frac{v_f}{2g}\ln\frac{1.95}{0.05} = 11.21\text{s} \ .$$

To obtain an expression for the velocity as a function of the traveled distance we can use expression (10) and write

$$s = \frac{1}{g}\int_0^v \frac{v\,dv}{1-(v/v_f)^2} = -\frac{v_f^2}{2g}\ln(1-(v/v_f)^2) \ .$$

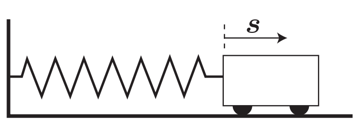
Solving for v we obtain

$$v = v_f\sqrt{1-e^{-2gs/v_f^2}}\ \text{m/s} \ .$$

We see that for, say, $v = 0.95v_f$, $s = 427.57$m. This is the distance traveled by the payload in 11.21s, which can be compared with the distance that would be traveled in the same time if we were to neglect air resistance, $s_{no\ drag} = gt^2/2 = 615.75$m.

Example ***Spring-mass system***

Here, we consider a mass allowed to move without friction on a horizontal slider and subject to the force exerted by a linear spring. Initially the system is in equilibrium (no force on the spring) at $s = 0$. Suddenly, the mass is given a velocity v_0 and then the system is left free to oscillate. We know that the effect of the spring is to cause an acceleration to the body, opposing the motion, of the form $a = -\kappa s$, where $\kappa > 0$ is a constant.



Using equation (7), we have

$$v^2 = v_0^2 - \kappa s^2 \ .$$

The displacement can now be obtained using expression (9),

$$t = \int_0^s \frac{ds}{\sqrt{v_0^2-\kappa s^2}} = \frac{1}{\sqrt{\kappa}}\arcsin\frac{\sqrt{\kappa}s}{v_0} \ ,$$

which gives,

$$s = \frac{v_0}{\sqrt{\kappa}} \sin \sqrt{\kappa} t \ .$$

Finally, the velocity as a function of time is simply, $v = v_0 \cos \sqrt{\kappa} t$. We recognize this motion as that of an undamped harmonic oscillator.

References

[1] R. Dugas, *A History of Mechanics*, Dover, 1988.

ADDITIONAL READING

J.L. Meriam and L.G. Kraige, *Engineering Mechanics, DYNAMICS*, 5th Edition

1/1, 1/2, 1/3, 1/4, 1/5 (Effect or Altitude only), 1/6, 1/7, 3/2