Control Volume (Integral) Technique

Introduction: Techniques for solving flow problems

There are three different techniques for solving these types of problems:

1. **Control Volume Analysis**
   - This is just like in thermodynamics class
     - One can calculate the **gross properties** (total power output, total heat transfer, etc.)
     - With this however, we do not care about the details inside the control volume (In other words we can treat the control volume as a "black box.")

2. **Differential Analysis**
   - In this technique, one solves differential equations of motion everywhere (i.e. The Navier-Stokes equation).
   - Here we solve for all of the details in the flow

3. **Dimensional Analysis and Experiment**
   - This method is used when methods one or two are not possible.
   - One uses wind tunnels, models, etc. to employ this method.

Introduction to systems and control volumes

**System:** Consider a system of a fluid:

- Recall from thermo class, that a **system** is defined as a volume of mass of fixed identity.
Let \( m \) = the mass of the system
Let \( \vec{V} \) = the velocity of the system
Let \( \vec{a} \) = the acceleration of the system

Now we can write three basic conservation laws which apply to this system. Note: These conservation laws apply directly to a system.

2. **Conservation of mass** states that the mass of a system is constant. This can be written as the following equation:

\[
\frac{d m_{\text{sys}}}{dt} = 0
\]

In this equation \( m \) = the mass of the system.

3. **Conservation of linear momentum** which is a restatement of Newton's Second Law.

   o In equation form this is written as:

   \[
   \sum F_{\text{sys}} = \frac{d}{dt} (m \vec{V})_{\text{sys}}
   \]

   Where \( m \vec{V} \) = the linear momentum of the system.

   o Note that this is the same as Newton's second law, it is just written a little differently.
Using the above equation, we can obtain a form of the equation in terms of acceleration:

\[
\sum F_{sys} = \frac{d}{dt} (mV)_{sys} = m \frac{dV}{dt} + V \frac{dm}{dt}
\]

This equation is obtained through expansion using the product rule.

We know that 
\[
\frac{dV}{dt} = a \quad \text{and} \quad \frac{dm}{dt} = 0
\]

by the conservation of mass found above.

Using these ideas, we can then see that

\[
\sum F_{sys} = m_{sys} a_{sys}
\]

4. **Conservation of Energy**

   For this, use the First Law of Thermodynamics in rate form to obtain the following equation:

   \[
   \frac{dE_{sys}}{dt} = Q_{sys} - W_{sys}
   \]

   Where \( E \) = the total energy of the system. In the above equation

   \[
   \frac{dE_{sys}}{dt}
   \]

   is the rate of change of system energy.
\[ Q_{\text{sys}} \] is the rate of heat added to the system 
\[ W_{\text{sys}} \] is the rate of work done by the system. Because work is done by the system, the negative sign is in the equation for the first law of thermodynamics.

Now, these conservation laws must always hold for a system.

5. **Conservation of Angular Momentum**

   We will not have time to study this, but see the text for details.

**Control Volume:**

- One can think of the system approach as the **Lagrangian description**, which if recalled is the description where we follow the individual chunks of the fluid.
- However, the trouble with this is that we prefer to use the **Eulerian Description**, where we define a control volume with fluid moving through it.

In other words, if we define some control volume, fixed in space, the system will flow through it.
Our goal here is to try to write the conservation laws above in a form applicable to a control volume (Eulerian description) rather than to a system (Lagrangian description).

There is a way to do this, and it is called the Reynolds Transport Theorem (R.T.T.)

Now our goal is to write all three of these conservation laws in terms of a control volume. But, to save us some work, let's not derive it three times, but just once for a general property B, and then use our result for any of the three conservation laws:

In other words, let's write R.T.T. for some arbitrary property B.

Let $B = \text{some arbitrary property (vector or scalar)}$ and $\beta = \frac{B}{m} \text{ per unit mass.}$

This way, we can later substitute B for mass, momentum, energy, etc.

We are then able to make the following substitutions for $\beta$:

- For conservation of mass, let $B = m$, and $\beta = \frac{m}{m} = 1$.
- For conservation of momentum, $B = \frac{mV}{m} = \frac{m}{m} = 1$.
- For conservation of energy, $B = E$, and $\beta = \frac{E}{m} = e$.

SEE the text for the derivation.

REYNOLDS TRANSPORT THEOREM (R.T.T.)

- This is for some property B, and is also for a fixed control volume. Deforming the control volume is more complicated; see the text for details.

The equation for Reynolds Transport Theorem is written as the following:

$$\frac{dB_{\text{sys}}}{dt} = \frac{d}{dt} \int \rho \beta dV + \int_{\text{cs}} \rho \beta (\nabla \cdot \mathbf{u}) dA$$

(This is equation 3.12 in the text.)

OR

For a fixed control volume, the order of differentiation and integration doesn't matter. So alternatively,
\[
\frac{dB_{sys}}{dt} = \int_{cv} \hat{C} (\rho \beta) d\mathcal{V} + \oint_{cs} \rho \beta (\nabla \cdot \mathbf{u}) d\mathcal{A} \\
\text{(This is equation 3.17 in the text.)}
\]

- **Comments** on the above equation:

  1. **The total rate of change of B following the system.**
     This is the term to which the conservation laws directly apply.
     \[
     \int_{cv} \frac{\partial}{\partial t} (\rho \beta) d\mathcal{V} \equiv
     \]

  2. The time rate of change of B within the control volume.
     This is due to unsteadiness.
     \[
     \oint_{cs} \rho \beta (\nabla \cdot \mathbf{u}) d\mathcal{A} \equiv
     \]

  3. The net flux of B out of the control surface.
     Due to **convection**, B changes because system moves to a new part of the flow field, where conditions are different. A circle was put around the integral to emphasize that this is an integral over the entire control surface. (This is not done in the text however).

  4. Sometimes \( \int_{cv} \) and \( \oint_{cs} \) are used to **emphasize** that these are volume and area integrals. (Again this is not done in the text).

Source:

http://www.mne.psu.edu/cimbala/Learning/Fluid/Control_Volume/index.htm