Complementary Energy

The linear elastic solid was considered in the previous section, with the characteristic straight force-deflection curve for axial deformations, Fig.8.2.2. Here, consider the more general case of a bar of *non-linear* elastic material, of length L, fixed at one end and subjected to a steadily increasing force P. The work dW done in extending the bar a small amount $d\Delta$ is

$$dW = Pd\Delta . \tag{8.3.1}$$

Force is now no longer proportional to extension Δ, Fig. 8.3.1. However, the total work done during the complete extension up to a final force P and final extension Δ is once again the total area beneath the force-extension curve. The work done is equal to the stored elastic strain energy which must now be expressed as an integral,

$$U = \int_0^\Delta P d\Delta \tag{8.3.2}$$

The strain energy can be calculated if the precise force-deflection relationship is known.

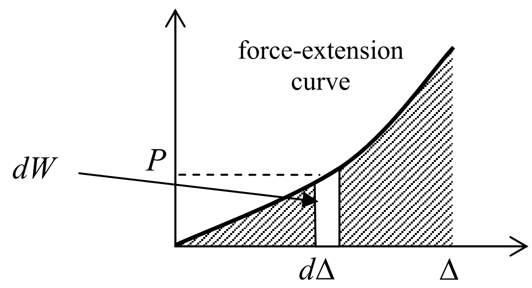

Figure 8.3.1: force-displacement curve for a non-linear material

8.3.1 Complementary Energy

The force-deflection curve is naturally divided into two regions, beneath the curve and above the curve, Fig. 8.3.2. The area of the region under the curve is the strain energy. It is helpful to introduce a new concept, the **complementary energy** C, which is the area above the curve; this can be seen to be given by

$$C = \int_0^P \Delta dP . \tag{8.3.3}$$

For a linear elastic material, $C = U$. Although C has units of energy, it has no real physical meaning.

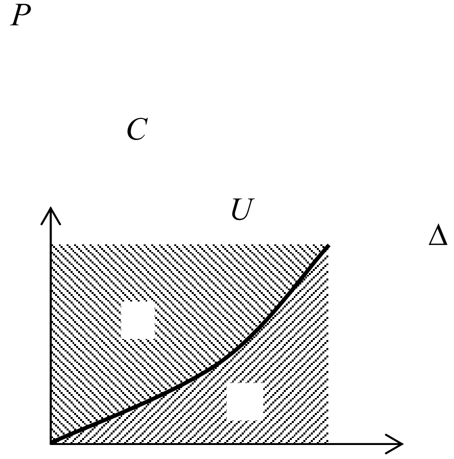

Figure 8.3.2: strain energy and complementary energy for an elastic material

8.3.2 The Crotti-Engesser Theorem

Suppose an elastic body is loaded by n independent loads $P_1, P_2, \ldots, P_n$. The strain energy is then the work done by these loads,

$$U = \int_0^{\Delta_1} P_1 d\Delta_1 + \int_0^{\Delta_2} P_2 d\Delta_2 + \cdots + \int_0^{\Delta_n} P_n d\Delta_n \qquad (8.3.4)$$

It follows that

$$\frac{\partial U}{\partial \Delta_j} = P_j \qquad (8.3.5)$$

which is known as Castigliano's first theorem.

Similarly, the total complementary energy is

$$C = \int_0^{P_1} \Delta_1 dP_1 + \int_0^{P_2} \Delta_2 dP_2 + \cdots + \int_0^{P_n} \Delta_n dP_n \qquad (8.3.6)$$

and it follows that

$$\frac{\partial C}{\partial P_j} = \Delta_j \qquad (8.3.7)$$

which is known as the **Crotti-Engesser theorem**. For a *linear* elastic material, $C = U$, and the Crotti-Engesser theorem reduces to Castigliano's second theorem, $\Delta_j = \partial U / \partial P_j$, Eqn. 8.2.25.