Comparison of Euler Theory with Experiment results

Limitations of Euler's Theory :

In practice the ideal conditions are never [i.e. the strut is initially straight and the end load being applied axially through centroid] reached. There is always some eccentricity and initial curvature present. These factors needs to be accommodated in the required formula's.

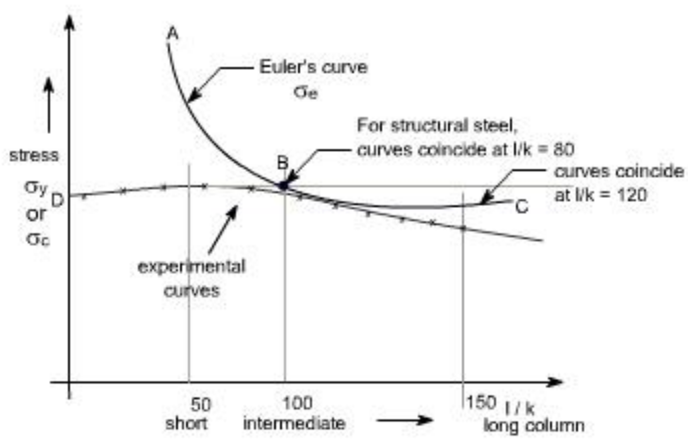
It is realized that, due to the above mentioned imperfections the strut will suffer a deflection which increases with load and consequently a bending moment is introduced which causes failure before the Euler's load is reached. Infact failure is by stress rather than by buckling and the deviation from the Euler value is more marked as the slenderness-ratio l/k is reduced. For values of l/k < 120 approx, the error in applying the Euler theory is too great to allow of its use. The stress to cause buckling from the Euler formula for the pin ended strut is

$$\text{Euler's stress, } \sigma_e = \frac{P_e}{A} = \frac{\pi^2 EI}{Al^2}$$

$$\text{But, } I = Ak^2$$

$$\sigma_e = \frac{\pi^2 E}{\left(\frac{l}{k}\right)^2}$$

A plot of σ_e versus l / k ratio is shown by the curve ABC.



Allowing for the imperfections of loading and strut, actual values at failure must lie within and below line CBD.

Other formulae have therefore been derived to attempt to obtain closer agreement between the actual failing load and the predicted value in this particular range of slenderness ratio i.e.l/k=40 to l/k=100.

(a) Straight – line formulae :

The permissible load is given by the formulae

$$P = \sigma_y A\left[1 - n\left(\frac{l}{k}\right)\right]$$ Where the value of index 'n' depends on the material used and the end conditions.

(b) Johnson parabolic formulae : The Johnson parabolic formulae is defined as

$$P = \sigma_y A\left[1 - b\left(\frac{l}{k}\right)^2\right]$$ where the value of index ‘b' depends on the end conditions.

(c) Rankine Gordon Formulae :

$$\frac{1}{P_R} = \frac{1}{P_e} + \frac{1}{P_c}$$

Where P_e = Euler crippling load

P_c = Crushing load or Yield point load in Compression

P_R = Actual load to cause failure or Rankine load

Since the Rankine formulae is a combination of the Euler and crushing load for a strut.

$$\frac{1}{P_R} = \frac{1}{P_e} + \frac{1}{P_c}$$

For a very short strut P_e is very large hence 1/ P_e would be large so that 1/ P_e can be neglected.

Thus $P_R = P_c$, for very large struts, P_e is very small so 1/ P_e would be large and 1/ P_c can be neglected ,hence $P_R = P_e$

The Rankine formulae is therefore valid for extreme values of 1/k.It is also found to be fairly accurate for the intermediate values in the range under consideration. Thus rewriting the formula in terms of stresses, we have

$$\frac{1}{\sigma A} = \frac{1}{\sigma_e A} + \frac{1}{\sigma_y A}$$

$$\frac{1}{\sigma} = \frac{1}{\sigma_e} + \frac{1}{\sigma_y}$$

$$\frac{1}{\sigma} = \frac{\sigma_e + \sigma_y}{\sigma_e . \sigma_y}$$

$$\sigma = \frac{\sigma_e . \sigma_y}{\sigma_e + \sigma_y} = \frac{\sigma_y}{1 + \frac{\sigma_y}{\sigma_e}}$$

For struts with both ends pinned

$$\sigma_e = \frac{\pi^2 E}{\left(\frac{l}{k}\right)^2}$$

$$\sigma = \frac{\sigma_y}{1 + \frac{\sigma_y}{\pi^2 E}\left(\frac{l}{k}\right)^2}$$

$$\sigma = \frac{\sigma_y}{1 + a\left(\frac{l}{k}\right)^2}$$

Where $a = \frac{\sigma_y}{\pi^2 EI}$ and the value of 'a' is found by conducting experiments on various materials. Theoretically, but having a value normally found by experiment for various materials. This will take into account other types of end conditions.

Therefore $$\text{Rankine load} = \frac{\sigma_y . A}{1 + a\left(\frac{l}{k}\right)^2}$$

Typical values of 'a' for use in Rankine formulae are given below in table.

Material	**σ_y or σ_c MN/m^2**	**Value of a**	
		Pinned ends	**Fixed ends**
Low carbon steel	315	1/7500	1/30000
Cast Iron	540	1/1600	1/64000
Timber	35	1/3000	1/12000

note a = 4 x (a for fixed ends)

Since the above values of 'a' are not exactly equal to the theoretical values , the Rankine loads for long struts will not be identical to those estimated by the Euler theory as estimated.

Strut with initial Curvature :

As we know that the true conditions are never realized , but there are always some imperfections. Let us say that the strut is having some initial curvature. i.e., it is not perfectly straight before loading. The situation will influence the stability. Let us analyze this effect.

by a differential calculus

$$R_0 \approx \frac{1}{d^2 y_0 / dx^2} \text{ (Approximately)}$$

Futher $\frac{E}{R} = \frac{M}{I}$ and $\frac{EI}{R} = M$

But for this case $EI\left[\frac{1}{R} - \frac{1}{R_0}\right] = M$

since strut is having some initial curvature

Now putting

$$\frac{1}{R} = \frac{d^2 y}{dx^2} \text{ and } \frac{1}{R_0} = \frac{d^2 y_0}{dx^2}$$

Where ' y_0' is the value of deflection before the load is applied to the strut when the load is applied to the strut the deflection increases to a value 'y'. Hence

$$EI\left[\frac{d^2 y}{dx^2} - \frac{d^2 y_0}{dx^2}\right] = M$$

$$EI\frac{d^2 y}{dx^2} - EI\frac{d^2 y_0}{dx^2} = M$$

$$EI\frac{d^2 y}{dx^2} = M + EI\frac{d^2 y_0}{dx^2}$$

$$EI\frac{d^2 y}{dx^2} = -Py + EI\frac{d^2 y_0}{dx^2}$$

If the pinended strut is under the action of a load P then obviously the BM would be' − py'

Hence

$$EI\frac{d^2 y}{dx^2} + Py = EI\frac{d^2 y_0}{dx^2}$$

$$\frac{d^2 y}{dx^2} + \frac{Py}{EI} = \frac{d^2 y_0}{dx^2}$$

Again letting

$$\frac{P}{EI} = n^2$$

$$\frac{d^2 y}{dx^2} + n^2 y = \frac{d^2 y_0}{dx^2}$$

The initial shape of the strut y_0 may be assumed circular, parabolic or sinusoidal without making much difference to the final results, but the most convenient form is

$y_0 = C.\sin\frac{\pi x}{l}$ where C is some constant or here it is amplitude

Which satisfies the end conditions and corresponds to a maximum deviation 'C'. Any other shape could be analyzed into a Fourier series of sine terms. Then

$$\frac{d^2y}{dx^2} + n^2y = \frac{d^2y_0}{dx^2} = \frac{d^2}{dx^2}\left[C.\sin\frac{\pi x}{l}\right] = \left(-C.\frac{\pi^2}{l^2}\right)\sin\left(\frac{\pi x}{l}\right)$$

The computer solution would be therefore be

$$y_{general} = y_{complementry} + y_{PI}$$

$$y = A\cos nx + B\sin nx + \frac{C.\frac{\pi^2}{l^2}}{\left(\frac{\pi^2}{l^2}\right) - n^2}\sin\left(\frac{\pi x}{l}\right)$$

Boundary conditions which are relevant to the problem are

at x = 0 ; y = 0 thus B = 0

Again

when x = l ; y = 0 or x = l / 2 ; dy/dx = 0

the above condition gives B = 0

Therefore the complete solution would be

$$y = \frac{C.\frac{\pi^2}{l^2}}{\left\{\left(\frac{\pi^2}{l^2}\right) - n^2\right\}} \sin\left(\frac{\pi x}{l}\right)$$

Again the above solution can be slightly rearranged. since

$$P_e = \frac{\pi^2 EI}{l^2}$$

hence the term $\frac{\frac{\pi^2}{l^2}}{\frac{\pi^2}{l^2} - n^2}$ after multiplying the denominator & numerator by EI is equal to

$$\frac{\frac{\pi^2 EI}{l^2}}{\frac{\pi^2 EI}{l^2} - n^2 EI} = \left[\frac{P_e}{P_e - P}\right]$$

Since $n^2 = \frac{P}{EI}$

where P_e = Euler's load P=applied load

Thus

$$y = \frac{C.\frac{\pi^2}{l^2}}{\left\{\left(\frac{\pi^2}{l^2}\right) - n^2\right\}} \sin\left(\frac{\pi x}{l}\right)$$

$$y = \left\{\frac{C.P_e}{P_e - P}\right\} \sin\left(\frac{\pi x}{l}\right)$$

The crippling load is again

$$P = P_e = \frac{\pi^2 EI}{l^2}$$

Since the BM for a pin ended strut at any point is given as

M = -Py and

Max BM = P y_{max}

Now in order to define the absolute value in terms of maximum amplitude let us use the symbol as ‘^'.

$$\hat{M} = P.\hat{y}$$

$$= C.\frac{P.P_e}{(P_e - p)}$$

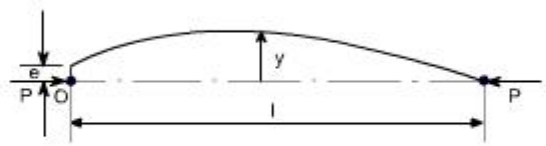
Therefore $\hat{M} = \frac{C\ P\ P_e}{[P_e - p]}$ since $y_{max^m} = \frac{P_e}{[P_e - p]}$

$\sin\frac{\pi x}{l} = 1$ when $\frac{\pi x}{l} = \frac{\pi}{2}$

Hence $\hat{M} = \frac{C\ P\ P_e}{[P_e - p]}$

Strut with eccentric load

Let 'e' be the eccentricity of the applied end load, and measuring y from the line of action of the load.



Then $E\,I\,\frac{d^2 y}{dx^2} = -P\,y$

or $(D^2 + n^2)\,y = 0$ where $n^2 = P / EI$

Therefore $y_{general} = y_{complementary}$

$= A\sin nx + B\cos nx$

applying the boundary conditions then we can determine the constants i.e.

at x = 0 ; y = e thus B = e

at x = l / 2 ; dy / dx = 0

Therefore

$$A\cos\frac{nl}{2} - B\sin\frac{nl}{2} = 0$$

$$A\cos\frac{nl}{2} = B\sin\frac{nl}{2}$$

$$A = B\tan\frac{nl}{2}$$

$$A = e\tan\frac{nl}{2}$$

Hence the complete solution becomes

$y = A\sin(nx) + B\cos(nx)$

substituting the values of A and B we get

$$y = e\left[\tan\frac{nl}{2}\sin nx + \cos nx\right]$$

Note that with an eccentric load, the strut deflects for all values of P, and not only for the critical value as was the case with an axially applied load. The deflection becomes infinite for tan (nl)/2 = ∞ i.e. nl = π giving the same crippling load $P_e = \frac{\pi^2 EI}{l^2}$. However, due to additional bending moment set up by deflection, the strut will always fail by compressive stress before Euler load is reached.

Since

$$y = e\left[\tan\frac{nl}{2}\sin nx + \cos nx\right]$$

$$y_{max^m}\Big|_{at\ x = \frac{l}{2}} = e\left[\tan\left(\frac{nl}{2}\right)\sin\frac{nl}{2} + \cos\frac{nl}{2}\right]$$

$$= e\left[\frac{\sin^2\frac{nl}{2} + \cos^2\frac{nl}{2}}{\cos\frac{nl}{2}}\right]$$

$$= e\left[\frac{1}{\cos\frac{nl}{2}}\right] = e\ \sec\frac{nl}{2}$$

Hence maximum bending moment would be

$$M_{max^m} = P\ y_{max^m}$$

$$= P\,e\ \sec\frac{nl}{2}$$

Now the maximum stress is obtained by combined and direct strain

$$\sigma = \frac{P}{A} + \frac{M}{Z} \text{ stress due to bending } \frac{\sigma}{y} = \frac{M}{I};$$

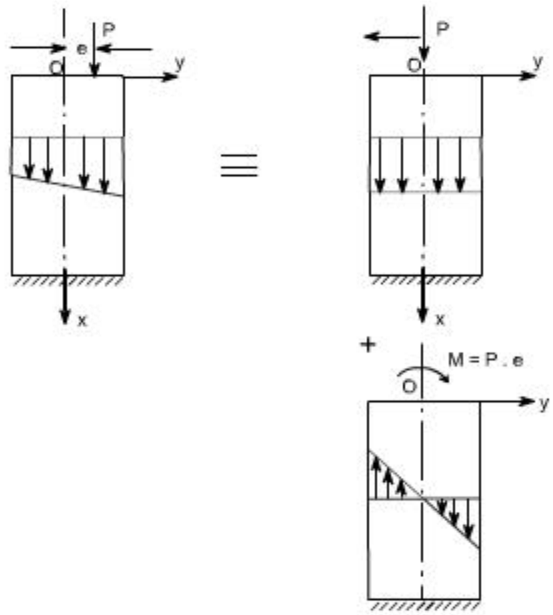
$$M = \sigma\frac{I}{y};\ \sigma_{max} = \frac{M}{Z} \text{ Wher } Z = I/y \text{ is section modulus}$$

The second term is obviously due the bending action.

Consider a short strut subjected to an eccentrically applied compressive force P at its upper end. If such a strut is comparatively short and stiff, the deflection due to bending action of the eccentric load will be neglible compared with eccentricity ‘e' and the principal of super-imposition applies.

If the strut is assumed to have a plane of symmetry (the xy - plane) and the load P lies in this plane at the distance ‘e' from the centroidal axis ox.

Then such a loading may be replaced by its statically equivalent of a centrally applied compressive force ‘P' and a couple of moment P.e



1. The centrally applied load P produces a uniform compressive $\sigma_1 = \frac{P}{A}$ stress over each cross-section as shown by the stress diagram.

2. The end moment 'M' produces a linearly varying bending stress $\sigma_2 = \frac{My}{I}$ as shown in the figure.

Then by super-impostion, the total compressive stress in any fibre due to combined bending and compression becomes,

$$\sigma = \frac{P}{A} + \frac{My}{I}$$

$$\sigma = \frac{P}{A} + \frac{M}{I/y}$$

$$\boxed{\sigma = \frac{P}{A} + \frac{M}{Z}}$$

Source: http://nptel.ac.in/courses/Webcourse-contents/IIT-ROORKEE/strength%20of%20materials/homepage.htm