CALIBRATION

Calibration of electronic nonautomatic weighing instruments - Error analysis

PASCHALIS TSIMITRAS, Kilkis, Greece

Summary

Various designs of nonautomatic electronic weighing instruments are employed with very different numbers of scale intervals. This paper introduces a new methodology which can be implemented in all designs and most specifically in single-range, multiple-range and multi-interval instruments.

This study is intended to serve the needs of users of weighing instruments who require confirmation of the accuracy of the weight values. The criteria to be satisfied are:

- Traceability to a national standard;
- Statement of uncertainty for the indicated (net weight) values without correction of systematic deviations; confidence level at least 95 % according to EAL-R2; and
- Consideration of the environmental conditions on the site at which the weighing is used during measurements.

1 Introduction

The proposed methodology aims at calculating the total uncertainty of the weighing instrument. More specifically, the total uncertainty is a function of both the random (precision) and the systematic (bias) uncertainty.

Considering a sub-case in which the random and the systematic uncertainties are not independent, the total uncertainty is the algebraic sum of the above-mentioned uncertainties.

The total uncertainty is based on the following parameters:

1 Repeatability
2 Resolution
3 Eccentricity
4 Deviations of indication - Linearity
5 Drift of instruments
6 Effect of convection
7 Standards weights and density of air
8 Hysteresis

2 Repeatability

The instrument should be set to zero before each measurement. The load should be placed on-center. A one-piece test load should preferably be used. For single-range instruments, the test load *P*, should be equal to *Max/2*. For multiple-range instruments,

$P = Max_i + (Max_{i+1} - Max_i)/2.$

The standard deviation, *s*, is calculated from the weight values, using:

$$s = \sqrt{\frac{1}{n-1} \cdot \sum_{i=1}^{n} \left(I_i - \bar{I}\right)^2} \qquad (1)$$

with

$$\bar{I} = \frac{1}{n} \cdot \sum_{i=1}^{n} I_i \qquad (2)$$

The standard uncertainty of the repeatability is calculated from:

$$u_w^2 = s^2 \qquad (3)$$

3 Resolution

The standard uncertainty of the resolution error of the indication, *I*, for diverse scale intervals d_i in multiple-range instruments is given by:

$$u_r^2 = \left(\frac{d_i}{2 \cdot \sqrt{3}}\right)^2 = \frac{d_i^2}{12} \qquad (4)$$

For single-range instruments, the variance of the rounding error is:

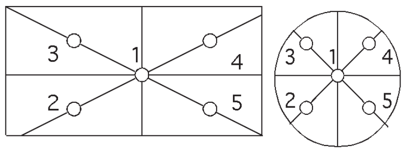
$$u_r^{\,2} = \left(\frac{d}{2\cdot\sqrt{3}}\right)^2 = \frac{d^2}{12} \qquad (5)$$

The assumption is that the distribution is rectangular. According to the rectangular distribution, the base is *d* and the height is 1/*d*.

4 Eccentric loading

The test load is applied at the positions shown below, which mark the center of gravity of the load for the appropriate measurement.

Central measurement $e_1 = 0$
Front left measurement e_2
Back left measurement e_3
Back right measurement e_4
Front right measurement e_5



After the first measurement, tare setting may be done when the instrument is loaded. A one-piece test load should preferably be used. For single-range instruments, the test load, *P*, should be equal to *Max/2*. For multiple-range instruments, $P = Max_i + (Max_{i+1} - Max_i)/2$.

4.1 Distribution of off-center load

An a-priori distribution is proposed, according to Figure 1.

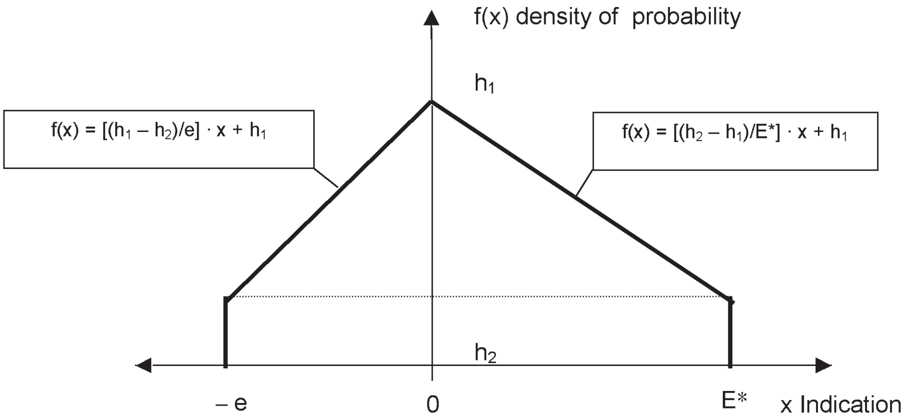


Fig. 1 A-priori distribution for eccentricity (at the center of the pan the density of probability is higher compared to out of center areas)

E* = the greatest positive difference between off-center and central loading indications

$E^* = \max(e_1, e_2, e_3, e_4, e_5)$ (6)

– e = the smallest negative difference between off-center and central loading indications

$-e = \min(e_1, e_2, e_3, e_4, e_5)$ (7)

$h_1 = \kappa h_2$ (8)

$$(E^*+e)\cdot h_2 + \frac{1}{2}\cdot(E^*+e)\cdot(h_1 - h_2) = 1 \Rightarrow h_2 = \frac{2}{(E^*+e)\cdot(\kappa+1)} \quad (9)$$

$$\int_{-\infty}^{\infty} x^2\cdot f(x)\cdot dx = \int_{-e}^{E^*} x^2\cdot f(x)\cdot dx = \int_{-e}^{0} x^2\cdot\left(\frac{h_1-h_2}{e}\cdot x + h_1\right)\cdot dx + \int_{0}^{E^*} x^2\cdot\left(\frac{h_2-h_1}{E^*}\cdot x + h_1\right)\cdot dx = \frac{\kappa+3}{6\cdot(\kappa+1)}\cdot\left(e^2 - E^*\cdot e + E^{*2}\right) \quad (10)$$

$$\bar{x} = \int_{-\infty}^{\infty} x\cdot f(x)\cdot dx = \int_{-e}^{E^*} x\cdot f(x)\cdot dx = \int_{-e}^{0} x\cdot\left(\frac{h_1-h_2}{e}\cdot x + h_1\right)\cdot dx + \int_{0}^{E^*} x\cdot\left(\frac{h_2-h_1}{E^*}\cdot x + h_1\right)\cdot dx = \frac{\kappa+2}{(\kappa+1)\cdot 3}\cdot(E^*-e) \quad (11)$$

$\bar{x}$ = mean average of the distribution

$$\sigma_{ecc}^2 = \int_{-\infty}^{\infty}(x-\bar{x})^2\cdot f(x)\cdot dx = \int_{-e}^{E^*} x^2\cdot f(x)\cdot dx - \bar{x}^2 = \frac{\kappa+3}{(\kappa+1)\cdot 6}\left(e^2 - e\cdot E^* + E^{*2}\right) - \left[\frac{\kappa+2}{(\kappa+1)\cdot 3}(E^*-\varepsilon)\right]^2 \quad (12)$$

E_{ecc} = the maximum value between E* and e

$$\sigma^2 = \theta \cdot E_{ecc}^2 \quad (13)$$

$$\kappa = (-5)\cdot\zeta + 25 \quad (14)$$

$$\zeta = \frac{|e_2| + |e_3| + |e_4| + |e_5|}{E_{ecc}} \quad (15)$$

For E* = e ⇒ symmetric distribution
and for $\kappa = 1 \Rightarrow \theta = 1/3$: rectangular A-priori distribution
and for $\kappa \to \infty \Rightarrow \theta = 1/6$: triangular A-priori distribution

$$E_{1ecc} = (1/2)\cdot(1/\lambda^2)\cdot E_{ecc}\cdot\lambda = E_{ecc}/(2\cdot\lambda) \quad (16)$$

With $\lambda = P_e / Max$

The variance v_{ecc} is given by:

$$v_{ecc} = \theta\cdot(E_{1ecc}/Max)^2 = \theta\cdot[E_{ecc}/(2\cdot\lambda\cdot Max)]^2 = \theta\cdot[E_{ecc}/(2\cdot P_e)]^2 \quad (17)$$

The standard uncertainty of eccentricity is given by:

$$u_{ecc}^2 = v_{ecc}\cdot I^2 \quad (18)$$

According to the assumption: $I \cong m_c$

5 Deviation of indication (Linearity)

5.1 Conventional weighing indication value

For the calculation of the error of indication, a new term is introduced: the conventional weighing indication value m_c^*, which is equal to the mass of a weight piece having a density ρ_c = 8000 [kg/m^3] at air density $\rho_{\alpha 0}$ = 1,2 [kg/m^3], and has the same weighing indication of a mass m having a density ρ_k at air density ρ_α.

$E = I - m_c^*$ (19)

E = Deviation of measurement

I = Indication of measurement

m_c^* = conventional indication value of standard weight

$$m_c^* \cdot \left(1 - \frac{\rho_{\alpha 0}}{\rho_c}\right) = m \cdot \left(1 - \frac{\rho_a}{\rho_k}\right) \Rightarrow m_c^* = m_c \cdot \frac{\rho_k - \rho_a}{\rho_k - \rho_{\alpha 0}} \quad (20)$$

and $m = m_c \cdot 0{,}99985 \cdot \rho_k / (\rho_k - 1{,}2)$ (21)

In the case $\rho_\alpha = \rho_{\alpha 0} \Rightarrow m_c^* = m_c$

m = mass

m_c = conventional value of mass of standard weight from calibration certificate

ρ_k = density of standard weight from calibration certificate [kg/m^3]

ρ_α = air density [kg/m^3]

$\rho_{\alpha 0}$ = 1,2 [kg/m^3]

ρ = 8000 [kg/m^3]

5.2 Evaluation

Measurement I	Conventional value of mass m_{ci} [g]	Conventional value of indication m_{ci}^* [g]	Indication [g] I_i	$I_i - m_{ci}^* = E_I$ [g]
1	Min	m_{c1}^*	I_1	E_1
2	$m_{c2} \approx (1/N) \cdot$ Max	m_{c2}^*	I_2	E_2
3	$m_{c3} \approx (2/N) \cdot$ Max	m_{c3}^*	I_3	E_3
4	m_{c4}	m_{c4}^*	I_4	E_4
...	...	...	...	...
N	$m_{cN} \approx$ Max	m_{cN}^*	I_N	E_N

If (I_1, E_1), ... (I_N, E_N) are the measured pairs of values, they are described by the linear equation $E = A + B \cdot I$, the values A_{best} and B_{best} result, which minimize the sum of the squares of the deviations.

$$A_{best} = \frac{\left(\sum I_i^2\right)\left(\sum E_i\right) - \left(\sum I_i\right)\left(\sum I_i \cdot E_i\right)}{\Delta} \quad (22)$$

$$B_{best} = \frac{N \cdot \left(\sum I_i \cdot E_i\right) - \left(\sum I_i\right)\left(\sum E_i\right)}{\Delta} \quad (23)$$

$$\Delta = N \cdot \left(\sum I_i^2\right) - \left(\sum I_i\right)^2 \quad (24)$$

$$\sigma_E^2 = \frac{1}{N-2} \cdot \sum_{i=1}^{N} \left(E_i - A_{best} - B_{best} \cdot I_i\right)^2 \quad (25)$$

where σ_E is the standard deviation of the straight line $A_{best} + B_{best} \cdot I$.

Additionally, the standard uncertainty for the parameters A_{best} and B_{best}, are:

$$\sigma_A^2 = \frac{\sigma_E^2 \cdot \sum I_i^2}{\Delta} \quad (26)$$

$$\sigma_B^2 = \frac{N \cdot \sigma_E^2}{\Delta} \quad (27)$$

and the systematic uncertainty is the greatest absolute value from:

$$\text{MAX}\ |A_{best} + B_{best} \cdot I_i \pm t_{95} \cdot \sigma_{linie}| \quad (28)$$

where t is the unilateral confidence level, which means that for a number of measurements N, the degree of freedom is N – 2.

$$\sigma_\varepsilon^2 = \left(\frac{\partial E}{\partial A} \cdot \sigma_A\right)^2 + \left(\frac{\partial E}{\partial B} \cdot \sigma_B\right)^2 = \sigma_A^2 + I^2 \cdot \sigma_B^2 \quad (29)$$

$$\sigma_{line}^2 = \left[\frac{\partial(A_{best} + B_{best} \cdot I)}{\partial E_1} \cdot \sigma_{\varepsilon 1}\right]^2 + \ldots + \left[\frac{\partial(A_{best} + B_{best} \cdot I)}{\partial E_N} \cdot \sigma_{\varepsilon N}\right]^2 = \frac{\sigma_\varepsilon^2}{N_\varepsilon} \quad (30)$$

with

$$\sigma_{\varepsilon 1} = \sigma_{\varepsilon 2} = \ldots = \sigma_{\varepsilon N} = \sigma_\varepsilon \quad (31)$$

and

$$N_\varepsilon = \frac{\Delta^2}{\tau_1 \cdot I^2 + \tau_2 \cdot I + \tau_3} \quad (32)$$

$$\tau_1 = \left(\sum I_i^2\right) \cdot N^2 - \left(\sum I_i\right)^2 \cdot N \quad (33)$$

$$\tau_2 = 2 \cdot \left(\sum I_i\right)^3 - 2 \cdot N \cdot \left(\sum I_i^2\right) \cdot \left(\sum I_i\right) \quad (34)$$

$$\tau_3 = N \cdot \left(\sum I_i^2\right) - \left(\sum I_i^2\right) \cdot \left(\sum I_i\right)^2 \quad (35)$$

$\max\{N_\varepsilon\} = N$ for $I = (\Sigma I_i)/N$

$$\sigma_{Em}^2 = \left[\frac{\partial(A_{best} + B_{best} \cdot I)}{\partial E_1} \cdot \sigma_{E1}\right]^2 + \ldots + \left[\frac{\partial(A_{best} + B_{best} \cdot I)}{\partial E_N} \cdot \sigma_{\varepsilon N}\right]^2 = \frac{\sigma_E^2}{N_\varepsilon} \quad (36)$$

with $\sigma_{E1} = \sigma_{E2} = \ldots = \sigma_{EN} = \sigma_E$ (37)

The calculation of the standard deviation $\sigma_{\sigma Em}$ of the average standard deviation σ_{Em} gives:

$$\sigma_{\sigma_{Em}} = \frac{1}{\sqrt{2\cdot(N-1)}}\cdot\sigma_{Em} \tag{38}$$

This aids in the evaluation of the standard deviation of the population through the evaluation of the standard deviation of the sample, which means that the confidence level of 99,75 % is less than:

$$u_E^2 = [\sigma_{Em} + t_{99,75}\cdot\sigma_{\sigma Em}]^2 \tag{39}$$

6 Uncertainty from drift of instruments

Considering:

$$\Delta t = t_{max} - t_{min} + U_t/2^{0,5} \tag{40}$$

as the change in temperature during calibration and:

U_t = the total uncertainty of the thermometer from its calibration certificate (with 2σ) according to the assumption $U_t = U_{t\,min} \cong U_{t\,max}$

TK = the effect of temperature on the mean gradient of the characteristic in ppm/K (estimate or data information sheet),

the variance v_t of the temperature effect, is calculated from:

$$v_t = (1/12)\cdot[\Delta t\cdot TK\cdot 10^{-6}/ppm]^2 \tag{41}$$

The assumption is that the distribution is rectangular. According to the rectangular distribution, the base is: $[\Delta t\cdot TK\cdot 10^{-6}/ppm]$ and the height: $1/[\Delta t\cdot TK\cdot 10^{-6}/ppm]$. The standard uncertainty of drift for the weighting instrument is:

$$u_t^2 = v_t\cdot I^2 \tag{42}$$

7 Effect of convection

Considering:

t_{air} = air temperature [°C] with total uncertainty U_{tair} (2σ)

$t_{weights}$ = standard weight temperature [°C] with total uncertainty $U_{tweights}$ (2σ)

$$\Delta t_{conv} = t_{weights} - t_{air} \pm [(U_{tair}^2 + U_{tweights}^2)^{0.5}]/2 \tag{43}$$

The relations between any of the quantities which have been referred to: Δt_{conv} *m* are non-linear, and their values are calculated according to the following equation - see [11]:

$$\Delta m_{conv} = -k_v m^{3/4}\frac{\Delta t_{conv}}{|\Delta t|^{1/4}} - k_h m\Delta t_{conv} \tag{44}$$

In the case where $\Delta t_{conv} > 0$

$k_v = 215 \cdot 10^{-9}$

$k_h = 75{,}4 \cdot 10^{-9}$

While for $\Delta t_{conv} < 0$

$k_v = 119 \cdot 10^{-9}$

$k_h = 20{,}2 \cdot 10^{-9}$

The standard uncertainty of the convection effect is calculated from:

$$u_{conv}^2 = \frac{\Delta m_{conv}^2}{12} \quad (45)$$

8 Uncertainty from standard weights and density of air

Air temperature, relative humidity and atmospheric pressure are measured, and the greatest and smallest values during calibration are recorded.

Thus for an air temperature between t_{min} and t_{max}, the standard uncertainty (1σ) is:

$$u_t^2 = \frac{(t_{max} - t_{min})^2}{12} + \frac{U_t^2}{2} \quad (46)$$

where U_t is the total uncertainty of the thermometer from the calibration certificate (with 2σ) according to the assumption $U_t = U_{t\,min} \cong U_{t\,max}$.

The same applies to the atmospheric pressure and the relative humidity:

$$u_p^2 = \frac{(p_{max} - p_{min})^2}{12} + \frac{U_p^2}{2} \quad (47)$$

$$u_{hr}^2 = \frac{(hr_{max} - hr_{min})^2}{12} + \frac{U_{hr}^2}{2} \quad (48)$$

Over the range of environmental conditions of 600 mbar ≤ p ≤ 1100 mbar, – 20 °C ≤ t ≤ + 40 °C and hr ≤ 80 %, the approximate formula, which deviates from the internationally recommended formula the value $\Delta\rho_a/\sigma_\alpha = 2\cdot 10^{-3}$, is:

$$\rho_\alpha = \frac{0{,}34848 \cdot p - 0{,}009024 \cdot hr \cdot e^{0{,}0612 \cdot t}}{273{,}15 + t} \quad (49)$$

where $p = (p_{max} + p_{min})/2$, $hr = (hr_{max} + hr_{min})/2$, $t = (t_{max} + t_{min})/2$ (50)

The relative uncertainty of the CIPM formula for the density of the air without the uncertainty of the measuring parameters, is $u_f/\sigma_a = 1 \cdot 10^{-4}$ (1σ).

The standard uncertainty (1σ) of air density is:

$$u_{\rho\alpha}^2 = \frac{(2 \cdot 10^{-3} \cdot \rho_\alpha)^2}{12} + (1 \cdot 10^{-4} \cdot \rho_\alpha)^2 + \left(\frac{\partial \rho_\alpha}{\partial p} \cdot u_p\right)^2 + \left(\frac{\partial \rho_\alpha}{\partial t} \cdot u_t\right)^2 + \left(\frac{\partial \rho_\alpha}{\partial hr} \cdot u_{hr}\right)^2 \quad (51)$$

where:

$$\frac{\partial \rho_\alpha}{\partial p} = \frac{0{,}34848}{273{,}15 + t} \tag{52}$$

$$\frac{\partial \rho_\alpha}{\partial hr} = \frac{-0{,}009024 \cdot e^{0{,}0612 \cdot t}}{273{,}15 + t} \tag{53}$$

and

$$\frac{\partial \rho_\alpha}{\partial t} = \frac{\left[0{,}009024 \cdot hr \cdot \left[1 - (273{,}15 + t) \cdot 0{,}0612\right] \cdot e^{0{,}0612 \cdot t} - 0{,}34848 \cdot p\right]}{(273{,}15 + t)^2} \tag{54}$$

In cases where ρ_{CIPM} is the calculated as a result from the CIPM formula of the density of the air, the standard uncertainty of the density of the air can be even lower, as follows:

$$u_{\rho\alpha}{}^2 = \frac{(\rho_\alpha - \rho_{CIPM})^2}{12} + \left(1 \cdot 10^{-4} \cdot \rho_\alpha\right)^2 + \left(\frac{\partial \rho_\alpha}{\partial p} \cdot u_p\right)^2 + \left(\frac{\partial \rho_\alpha}{\partial t} \cdot u_t\right)^2 + \left(\frac{\partial \rho_\alpha}{\partial hr} \cdot u_{hr}\right)^2 \tag{55}$$

The standard uncertainty of the conventional indication is:

$$u_{mc^*}{}^2 = \left(\frac{\partial m_c{}^*}{\partial m_k} \cdot u_{mk}\right)^2 + \left(\frac{\partial m_c{}^*}{\partial \rho_\alpha} \cdot u_{\rho\alpha}\right)^2 + \left(\frac{\partial m_c{}^*}{\partial \rho_k} \cdot u_{\rho k}\right)^2 \tag{56}$$

where $u_{\rho k}$ = the standard uncertainty (1σ) of the density of the standard weights [kg/m^3] from the calibration certificate.

The variable which refers to standard weights and the air density, is calculated as follows:

$$v_k = \left(\frac{\rho_k - \rho_\alpha}{\rho_k - \rho_{\alpha 0}} \cdot \frac{\sum U_{Di}}{k \cdot m_{c0}}\right)^2 + \left(\frac{1}{\rho_k - \rho_{\alpha 0}} \cdot u_{\rho\alpha}\right)^2 + \left(\frac{\rho_\alpha - \rho_{\alpha 0}}{(\rho_k - \rho_{\alpha 0})^2} \cdot u_{\rho k}\right)^2 \tag{57}$$

$$u_{mc^*}{}^2 = v_k \cdot I^2 \tag{58}$$

According to the assumption: $I \cong m_{ci}$

ΣU_i = Uncertainty of the standard weight (2σ) from the calibration certificate

$\Sigma U_{Di} = k_D \cdot \Sigma U_i$, $1 \le k_D \le 3$, k_D *Drift*, where k_D is the quantitative coefficient of the drift of the standard weight

$k = 2$

m_{C0} = conventional mass from the calibration certificate of the weight ≅ Max value of weighing instrument.

9 Hysteresis

The test loads P_i, tare values TL_i and indications I_i were chosen or determined as below. Total uncertainty during unloading of the weighing instrument is the same as during loading. The calculation of random and systematic uncertainty is similar to that in paragraph 5.

Measurement i	Tare values TL_i	Load	Conventional value of mass m_{ci} [g]	Conventional value of indication $m_{Ci}{}^*$ [g]	Indication [g] I_i	$I_i - m_{ci}{}^* = E_I$ [g]
1	≈ Max	≈ (1/N)Max	m_{c1}	$m_{c1}{}^*$	I_1	E1↓
2	≈ Max	≈ (2/N)Max	m_{c2}	$m_{c2}{}^*$	I_2	E2↓
…	≈ Max	…	…	…	…	…
N–1	≈ Max	≈ [(N–1)/N]Max	m_{cN-1}	$m_{cN-1}{}^*$	I_{N-1}	EN-1↓
N	≈ Max	≈ Max	m_{cN}	$m_{cN}{}^*$	I_N	EN↓

10 Total uncertainty of measurement

The effective degrees of freedom from the Welch-Satterthwaite formula, is:

$$v_{eff} = \frac{u_c^4}{\sum \frac{u_i^4}{v_i}} \qquad (59)$$

where u_c is the combined standard uncertainty (1σ).

$$u_c = \sqrt{u_w^2 + u_r^2 + u_E^2 + u_{conv}^2 + (v_{ecc} + v_t + v_k)I^2} \qquad (60)$$

The coverage factor t_p, is calculated according to the following formula:

$$t_p = k_p \cdot \sqrt{1 + \frac{2}{v_{eff}}} \qquad (61)$$

where $k_p = 2$ (62)

The uncertainty of measurement comprises type A and type B components. For multiple range instruments, the formula is applied to each range, separately. The formula for total uncertainty (2σ) is:

$$U_{total} = \underbrace{t_p \cdot \sqrt{u_w^2 + u_r^2 + u_E^2 + u_{conv}^2 + (v_{ecc} + v_t + v_k) \cdot I^2}}_{U_{random}} + \underbrace{\max\left| A_{best} + B_{best} \cdot I \pm t_{95} \cdot \frac{\sigma_\varepsilon}{\sqrt{N_\varepsilon}} \right|}_{U_{systematic}} \qquad (63)$$

Total uncertainty during loading (↑)and unloading (↓) of the weighing instrument, is:

$$U_{total} \uparrow\downarrow = \sqrt{U_{random}^2 \uparrow + U_{random}^2 \downarrow} + \left| U_{systematic} \uparrow + U_{systematic} \downarrow \right| \qquad (64)$$

where stochastic parts of the systematic uncertainties are geometrically added.

11 Determination of mass

In cases where the mass m_t must be calculated, considering an object with density ρ_t, standard uncertainty of density $u_{\rho t}$ (1σ) and air density $\rho_{\alpha t}$ we have measurement on the indication W_t (total uncertainty of weighing instrument U_{wt}) of the weighting instrument, the mass is:

$$m_t = \frac{0{,}99985 \cdot W_t \cdot \rho_t}{\rho_t - \rho_{\alpha t}} \qquad (65)$$

while the calculated total uncertainty of the object U_t is calculated by the formula:

$$U_t = 2 \cdot \left\{ \left[\frac{0{,}99985 \cdot \rho_t}{\rho_t - \rho_{\alpha t}} \cdot \frac{U_{wt}}{2} \right]^2 + \left[\frac{-0{,}99985 \cdot W_t \cdot \rho_{\alpha t}}{(\rho_t - \rho_{\alpha t})^2} \cdot u_{\rho t} \right]^2 + \left[\frac{-0{,}99985 \cdot W_t \cdot \rho_t}{(\rho_t - \rho_{\alpha t})^2} \cdot u_{\rho \alpha t} \right]^2 \right\}^{1/2} \qquad (66)$$

12 Examples

12.1 Single-range instrument

The instrument characteristics are: Max = 320 g, d = 0,001 g

12.2 Environmental conditions

	Min	Max	Mean	Total uncertainty (of instruments) (2σ)	Standard uncertainty (1σ)
Air pressure (mbar)	962,7	962,9	962,8	0,22	u_p = 0,18
Air temperature (°C)	17,3	17,9	17,6	0,3	u_t = 0,27
Relative humidity (%)	40	43	41,5	3	u_{hr} = 2,29

Density of air from formula (47): $\rho_a = 1{,}1502$ [kg/m³]

Density of air from the CIPM formula: $\rho_{CIPM} = 1{,}150175$ [kg/m³]

$(\partial \rho_a / \partial p) = 0{,}0012$ [kg/m³] / [mbar]

$(\partial \rho_a / \partial t) = -\,0{,}0042$ [kg/m³] / [°C]

$(\partial \rho_a / \partial hr) = -\,9{,}06 \cdot 10^{-5}$ [kg/m³] / [%]

$$u_{\rho a}^2 = [(\rho_a - \rho_{CIPM})^2/12] + (1 \cdot 10^{-4} \cdot \rho_a)^2 + [(\partial \rho_a/\partial p) \cdot u_p]^2 + [(\partial \rho_a/\partial t) \cdot u_t]^2 + [(\partial \rho_a/\partial hr) \cdot u_{hr}]^2$$

$$u_{\rho a}^2 = 0{,}01 \cdot 10^{-9} + 13{,}23 \cdot 10^{-9} + 39{,}55 \cdot 10^{-9} + 1315{,}07 \cdot 10^{-9} + 43{,}60 \cdot 10^{-9}$$

$$u_{\rho a}^2 = 1{,}41 \cdot 10^{-6}$$

$$u_{\rho a} = 0{,}0012 \text{ [kg/m}^3\text{]}$$

12.3 Repeatability

P = 100 g is chosen as the test load. The readings in the table at the top of page 15 were recorded.

Measurement i	Indication [g]
1	100,000
2	100,001
3	100,000
4	100,000
5	100,000
6	100,001

This yields:

Standard deviation s = 0,000516 [g]

$u_w = s^2 = 26,67 \cdot 10^{-8}$ [g^2]

12.4 Resolution

The variance of the rounding error is:

$u_r^2 = [(d/2) \cdot 3^{-0.5}]^2 = d^2/12 = 8,33 \cdot 10^{-8}$ [g^2]·

12.5 Eccentricity (Off-center loading)

P = 200 g was chosen as the test load. The following readings were recorded:

200,000 g, tared 0 g
$e_2 = 0,001$ [g]
$e_3 = 0,000$ [g]
$e_4 = -0,002$ [g]
$e_5 = 0,003$ [g]

This yields:

e = 0,002 [g]
E* = 0,003 [g]
$\zeta = 2$
$\kappa = 15$
$\sigma_{ecc}^2 = 1,187 \cdot 10^{-6}$
$E_{ecc} = 0,003$ [g]
$\theta = 0,132$
$v_{ecc} = 7,42 \cdot 10^{-12}$
$u_{ecc}^2 = v_{ecc} \cdot I^2$

12.6 Deviation of indication (Linearity)

The test loads and indications, I_i, were chosen or determined as follows:

Measurement i	Conventional value of mass m_{ci} [g]	Conventional value of indication m^*_{Ci} [g]	Indication [g] I_i	$I_i - m_{ci}^* = E_I$ [g]
1	0,02001	0,02001	0,020	0,0000
2	39,99997	40,00022	40,000	– 0,0002
3	80,00012	80,00062	80,000	– 0,0006
4	120,00012	120,00087	120,000	– 0,0009
5	160,00016	160,00116	160,000	– 0,0012
6	200,00018	200,00143	200,000	– 0,0014
7	240,00020	240,00169	240,000	– 0,0017
8	280,00030	280,00204	280,001	– 0,0010
9	320,00030	320,00229	320,001	– 0,0013

Standards weights of class E_2 with density ρ_{ki} = 8000 [kg/m^3] and standard uncertainty of density $u_{\rho ki}$ [1σ] = 100 [kg/m^3], are selected.

A_{best} = – 0,00024 [g] B_{best} = – 4,30·10^{-6} [g/g] Δ = 863947,44 [g^2]

σ_E^2 = 10,07 · 10^{-8} [g^2] σ_A^2 = 3,81 · 10^{-8} [g^2] σ_B^2 = 1,05 · 10^{-12} [g^{2}/g^2]

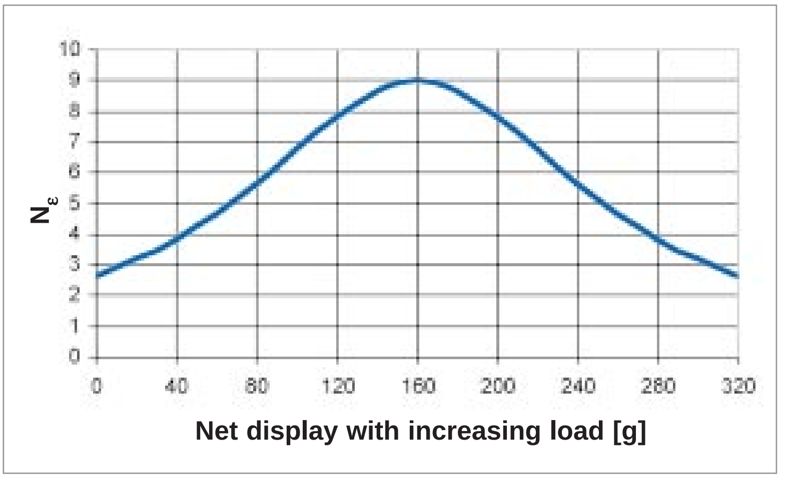
The systematic error is the greatest absolute value from:

$$\text{MAX}\ |\ A_{best} + B_{best} \cdot I_i \pm (t_{95}/N_\varepsilon^{1/2}) \cdot [\sigma_A^2 + I^2\ \sigma_B^2]^{0,5}\ | =$$

$$= 0{,}00024 + (4{,}30 \cdot 10^{-6}) \cdot I + (1{,}89/N_\varepsilon^{1/2}) \cdot [3{,}81 \cdot 10^{-8} + (1{,}05 \cdot 10^{-12}) \cdot I^2]^{0,5}$$

where t_{95} corresponds to a unilateral confidence level of 95 % (see DIN1319-3).

Fig. 2 Relationship between N_ε and indication (max = N for I = (ΣI_i)/N)



$$u_E^{\ 2} = \left[\left(\frac{\sqrt{10{,}07\cdot 10^{-8}}}{\sqrt{\dfrac{863947{,}44^2}{7775526{,}99\cdot I^2 - 2488206650\cdot I + 2{,}81993\cdot 10^{11}}}}\right)\cdot\left(1+\frac{4{,}32}{\sqrt{2\cdot(9-1)}}\right)\right]^2$$

12.7 Uncertainty from drift of instruments

$\Delta t = t_{max} - t_{min} + U_t/2^{0,5} = 0{,}81$ [°C]

TK = 2 ppm

$v_{TK} = (1/12) \cdot [\Delta t \cdot TK \cdot 10^{-6}/ppm]^2 = 0{,}22 \cdot 10^{-12}$

$u_{TK}^2 = v_{TK} \cdot I^2$

12.8 Effect of convection

$\Delta t_{conv} = (t_{weights} - t_{air}) + [(U_{tair}^2 + U_{tweights}^2)^{0,5}]/2 = (20{,}4 - 17{,}5) + [(0{,}3^2 + 0{,}2^2)^{0,5}]/2 = 3{,}08$ [°C]

$$u_{conv}^{\ 2} = \left[\frac{215\cdot 10^{-9}\cdot I^{3/4}\cdot 3{,}08^{3/4} + 75{,}4\cdot 10^{-9}\cdot I\cdot 3{,}08}{\sqrt{12}}\right]^2$$

12.9 Uncertainty from standard weights and density of air

$u_{mc*}^2 = v_k \cdot I^2$

$k_D = 1{,}5 \quad k = 2$

$\Sigma U_i = 0{,}175$ [mg] = 0,000175[g]

$\Sigma U_i = 0{,}0002625$ [g]

$v_k = 0{,}20 \cdot 10^{-12}$

12.10 Total uncertainty

The total uncertainty is calculated according to the following formula:

$U = t_p \cdot \{26{,}67 \cdot 10^{-8} + 8{,}33 \cdot 10^{-8} + 7{,}42 \cdot 10^{-12} \cdot I^2 +$

$$+\left[\left(\frac{\sqrt{10{,}07\cdot 10^{-8}}}{\sqrt{\dfrac{863947{,}44^2}{7775526{,}99\cdot I^2 - 2488206650\cdot I + 2{,}81993\cdot 10^{11}}}}\right)\cdot\left(1+\frac{4{,}32}{\sqrt{2\cdot(9-1)}}\right)\right]^2 +$$

$$+ 0{,}22\cdot 10^{-12}\cdot I^2 + \left[\frac{215\cdot 10^{-9}\cdot I^{3/4}\cdot 3{,}08^{3/4} + 75{,}4\cdot 10^{-9}\cdot I\cdot 3{,}08}{\sqrt{12}}\right]^2 + 0{,}20\cdot 10^{-12}\cdot I^2 \ \}^{1/2} +$$

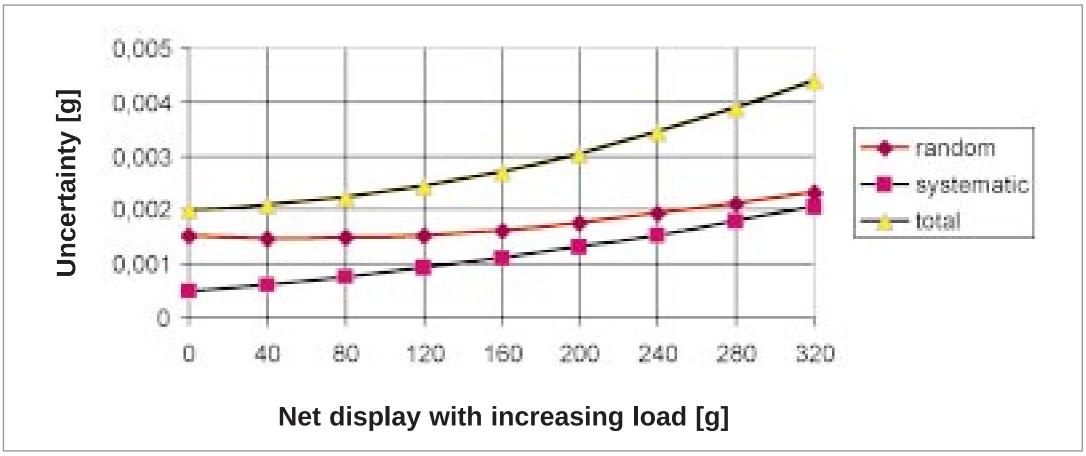
$$+ 0{,}00024 + 4{,}30\cdot 10^{-6}\cdot I + \frac{1{,}89}{\sqrt{\dfrac{863947{,}44^2}{7775526{,}99\cdot I^2 - 2488206650\cdot I + 2{,}81993\cdot 10^{11}}}}\cdot\left[3{,}81\cdot 10^{-8} + 1{,}05\cdot 10^{-12}\cdot I^2\right]^{1/2}$$

The total uncertainty using the approximate formula is:

$U_{total} = (-1 \cdot 10^{-13}) \cdot I^4 + (6 \cdot 10^{-11}) \cdot I^3 + (8 \cdot 10^{-9}) \cdot I^2 + (2 \cdot 10^{-6}) \cdot I + 0{,}0002$

with $R^2 = 1$

Fig. 3 Relationship between indication and uncertainties



12.11 Uncertainty budget

Test	Distribution	fd	I = 320 [g]		I = 160 [g]		I = 80 [g]	
			$u^2[1\sigma][g^2]$	$100{*}u_i/u_c$	$u^2[1\sigma][g^2]$	$100{*}u_i/u_c$	$u^2[1\sigma][g^2]$	$100{*}u_i/u_c$
Repeatability	Student	5	$26{,}67 \cdot 10^{-8}$	45,0 %	$26{,}67 \cdot 10^{-8}$	66,7 %	$26{,}67 \cdot 10^{-8}$	74,7 %
Resolution	Rectangular	∞	$8{,}33 \cdot 10^{-8}$	25,1 %	$8{,}33 \cdot 10^{-8}$	37,3 %	$8{,}33 \cdot 10^{-8}$	41,8 %
Eccentricity	“New”	∞	$75{,}97 \cdot 10^{-8}$	75,9 %	$18{,}99 \cdot 10^{-8}$	56,3 %	$4{,}74 \cdot 10^{-8}$	31,5 %
Deviations of indication-linearity	Gaussian	∞	$16{,}46 \cdot 10^{-8}$	35,3 %	$4{,}84 \cdot 10^{-8}$	28,4 %	$7{,}75 \cdot 10^{-8}$	40,3 %
Uncertainty from drift of instruments	Rectangular	∞	$2{,}25 \cdot 10^{-8}$	13,1 %	$0{,}56 \cdot 10^{-8}$	9,7 %	$0{,}14 \cdot 10^{-8}$	5,4 %
Effect of convection	Rectangular	∞	$0{,}10 \cdot 10^{-8}$	2,8 %	$0{,}03 \cdot 10^{-8}$	2,2 %	$0{,}01 \cdot 10^{-8}$	1,3 %
Uncertainty from standard weights and density of air	Gaussian	∞	$2{,}01 \cdot 10^{-8}$	12,4 %	$0{,}50 \cdot 10^{-8}$	9,2 %	$0{,}13 \cdot 10^{-8}$	5,1 %
			$131{,}80 \cdot 10^{-8}$		$59{,}93 \cdot 10^{-8}$		$47{,}77 \cdot 10^{-8}$	
	u_c		$1{,}15 \cdot 10^{-3}$		$0{,}77 \cdot 10^{-3}$		$0{,}69 \cdot 10^{-3}$	
	$t_p(v)$		2,020		2,078		2,121	
Random uncertainty			$2{,}31 \cdot 10^{-3}$ g		$1{,}61 \cdot 10^{-3}$ g		$1{,}47 \cdot 10^{-3}$ g	
Systematic uncertainty			$2{,}06 \cdot 10^{-3}$ g		$1{,}09 \cdot 10^{-3}$ g		$0{,}75 \cdot 10^{-3}$ g	
Total uncertainty			0,0044 g		0,0027 g		0,0022 g	

13 Conclusions

A new a-priori distribution has been introduced for eccentricity, where the coefficient κ is determined according to the characteristics of eccentric loading, for each weighing instrument.

The minimum value of random uncertainty is not found for I = 0 (in the paradigm of the current paper the minimum value is found for I = 59 g). As "N" increases the minimum random uncertainty takes a smaller value and this minimum is transferred to higher indications.

The formulation of the systematic error as $A_{best} + B_{best}$ I, gives the most probable value of the population but not for a confidence level of at least 95 %. Additionally, the formulation:

$$\max\left|A_{best} + B_{best} \cdot I \pm t_{95} \cdot \sqrt{\frac{\sigma_A^{\ 2} + I^2 \cdot \sigma_B^{\ 2}}{N_\varepsilon}}\right|$$

determines the highest level, so that a statistic hypothesis can be made that the systematic uncertainty of the population with a possibility of 95 % is smaller than the aforementioned highest limit.

The population is defined as the number of scale intervals, the quotient Max_i/d_i of the maximum capacity of each partial range and the appropriate scale interval (at this article's paradigm it is considered as 320 000). ■

References

[1] DIN V ENV 13005 Leitfaden zur Angabe der Unsicherheit beim Messen 1999. German version of the "Guide to the expression of uncertainty in measurement"

[2] Kochsiek,M.: Gläser M.1997 Massebestimmung-VCH Weinheim

[3] Kochsiek,M. Grundlagen der Massebestimmung 10/92 Mettler Toledo AG

[4] Taylor John R. Fehleranalyse/hrsg. 1988 – VCH Weinheim. German version of "An Introduction to Error Analysis"

[5] Archimedes; De architectura libri decem from Marcus Vitruvius Pollio

[6] DIN 1319-3 Grundlagen der Meßtechnik. Teil 3: Auswertung von Messungen einer einzelnen Meßgröße Meßunsicherheit

[7] DKD-3 Angabe der Meßunsicherheit bei Kalibrierungen. German version of EAL–R2 "Expression of the Uncertainty of Measurement in Calibration"

[8] OIML R 111 Weights of Classes E_1, E_2, F_1, F_2, M_1, M_2, M_3

[9] EN 45501: Metrological Aspects of Nonautomatic Weighing Instruments

[10] Einführung zur Fehlerrechnung im Praktikum / P. Blüm - Universität Karlsruhe, 2001

[11] M. Gläser -Change of the apparent mass of weights arising from temperature differences / Metrologia 36 (1999) pp. 183-197

[12] DKD-R7-1 Kalibrierung elektronischer nichtselbsttätiger Waagen Blatt 1, 2, 3 (1998)

[13] Weise, K.; Wöger, W. 1999 Meßunsicherheit und Meßdatenauswertung. Wiley-VCH, Weinheim

[14] Giacomo, P.; Equation for the Determination of the Density of Moist Air. Metrologia 18 (1982) pp. 33-40

[15] Davis,R.S.; Equation for the Determination of the Density of Moist Air. (1981/91) Metrologia 29 (1992) pp. 67-70

Acknowledgement

The author wishes to acknowledge the kind assistance of Dr. Kleanthis T. Geramanis.

Paschalis Tsimitras

tsimitras.paschalis@vdi.de