On the Genus of Pancake Network

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Abstract: *Both the pancake graph and star graph are Cayley graphs and are especially attractive for parallel processing. They both have sublogarithmic diameter, and are fairly sparse compared to hypercubes. In this paper, we focus on another important property, namely the genus. The genus of a graph is the minimum number of handles needed for drawing the graph on the plane without edges crossing. We will investigate the upper bound and lower bound for the genus of pancake graph and compare these values with the genus of the star graph as well as that of the hypercube.*

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1. Introduction

Sorting a sequence S of n elements by prefix reversal is a sorting technique in which the only operation allowed is a flipping of some prefix (of length two or more) of the sequence S. A prefix reversal of size *i* or an *i-flip* action is defined by $[p_1, p_2, \ldots, p_i, p_{i+1}, \ldots, p_n] \rightarrow [p_i, \ldots, p_2, p_1, p_{i+1}, \ldots, p_n]$. For example [5, 3, 2, 6, 1, 4] → [6, 2, 3, 5, 1, 4] is a 4-flip [12]. There are (n-1) possible flips since there are n -1 prefixes of length two or more. For sorting a sequence of n elements, Gates and Papadimitriou [7] showed that the number of flips is at least $\frac{17}{16}n$ and at most $\frac{5n+5}{3}n$. The set of all possible permutations form a symmetric group where the (n-1) flips form its generator. Akers et al. [1] introduced an interconnection network where each vertex represents an element of the symmetric group. Two vertices are linked by an edge if and only if their corresponding permutations are obtained from one another by a prefix reversal. This is related to the problem posed by Dweighter [6] which he states as follows. A chef prepares a stack of pancakes of different sizes. A waiter wanted to arranged the stack from smallest at the bottom to the largest at the top. He is only allowed to flip the top k pancakes, $1 < k \leq n$, as many times as is required to achieve the desired arrangement. If the pancake are burnt on one side, the desired arrangement would have the added requirement that the burnt side facing down. This is called the burnt-pancake problem.

The pancake graph $P_n = (S_n, E_n)$ of dimension n is defined as follows: S_n = {P| P is a permutation of {1, 2, …, n}} and E_n = {(P, Q)| P → Q is a prefix reversal of size i, $1 < i \leq n$}. The pancake graphs P_2 and P_3 are depicted in Figure 1(a) and (b), respectively. Figure 2 shows the pancake graph P_4. P_n is a vertex symmetric graph that has *n!* vertices since there are *n!* permutations of {1, 2, …, n}. As there are *n-1* prefix reversals that can be applied on a permutation (from 2-flip to n-flip), P_n is a regular graph with degree n-1, and has $n!(n-1)/2$ edges. P_n is also Hamiltonian although it is not bipartite [1]. The diameter of P_n ranges from *(15/14)n* to *(18/11)n* [5, 9] and was verified (by computed values for n up to *17*) to be smaller than the diameter of the star graph [2, 13].

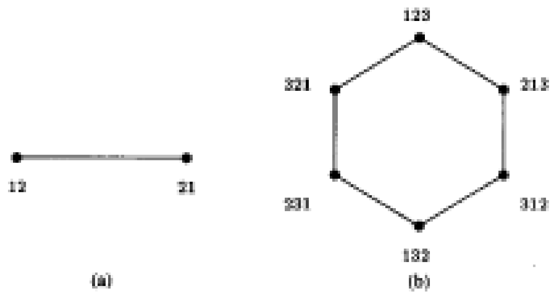


Figure 1. Pancake graphs (a) P_2, and (b) P_3.

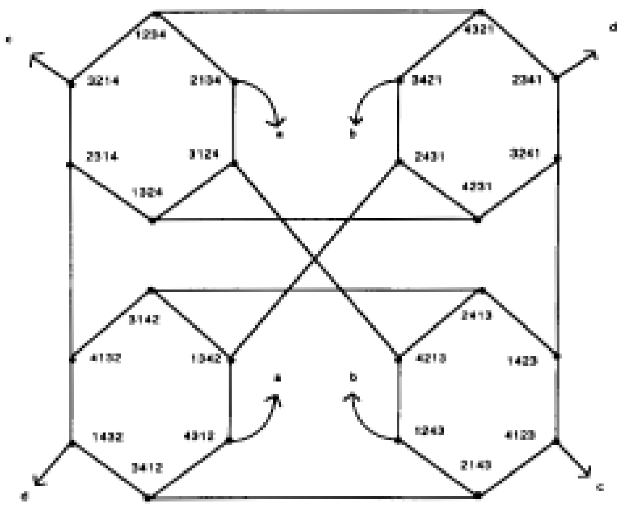


Figure 2. Pancake graph P_4.

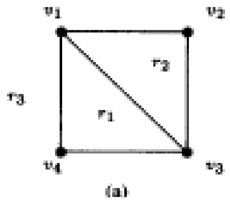
While the degree and diameter of the hypercube are logarithmic with respect to the network size, the degree and diameter of the pancake graph are, like those of the star graph, sub-logarithmic. The pancake graph of dimension n, P_n, is constructed recursively from n copies of P_{n-1}, by assigning a different element from the set {1, 2, …, n}as a suffix to each copy. Vertices of each of these n copies are permutations that have the same last element. Fig. 2 shows 4 copies of P_3 in P_4. The top right P_3 contains permutations whose last element is "1". For easier reference, we denote this P_3 as $P_{3,1}$. The top left P_3 contains permutations ending with "4", we denote this P_3 as $P_{3,4}$, and so on.

2. Preliminaries

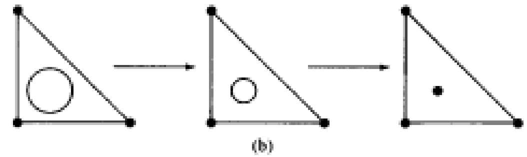
A pancake graph is an attractive model for interconnection networks. It is a better alternative to the most popular interconnection network model, namely the hypercube. While both the hypercube and the pancake are Cayley graphs, and both are vertex symmetric, the degree of every vertex in the pancake as well as its diameter are sub-logarithmic whereas the hypercube has logarithmic degree and diameter as functions of the number of vertices . The pancake graph has been studied extensively in the literature [1, 2, 9, 12, 13].

The pancake graph is a powerful modeling tool for various real life problems from parallel computing to biological engineering. For example, In [8], the authors investigated the use of the burnt pancake problem by considering segments of DNA of E-Coli as the pancakes. Flips were driven by genes from injection of salmonella typhimurium. They found out that E-Coli cells become antibiotic resistant when those segments of DNA are properly sorted. The time required to reach the mathematical solution in the bugs reflects the minimum number of flips needed to solve the burnt pancake problem [10].

A graph G can be embedded with no edge crossings on a compact orientable two-manifold surface on which a number of handles have been placed. A handle is used by an edge that would otherwise intersect with other edges at a point other than the endpoints. The genus of a surface is the number of handles on that surface. The genus of a graph G, denoted $\gamma(G)$, is the minimum genus of all possible surfaces on which G can be embedded. Planar graphs have genus zero since no handles are needed to prevent edge crossings. As a measure of the complexity of a network, the genus gives an indication of how efficiently the network can be laid out. The smaller the genus is, the more efficient the layout is. A region of a graph G embedded on some surface is the connected section of the surface bounded by a set of edges of G. This set of edges is the boundary of the region that uniquely defines the region. In Figure 3(a) there are three regions: r_1, r_2, and r_3. Region r_1 is bounded by edges (v_1, v_3), (v_3, v_4), and (v_4, v_1); region r_2 by edges (v_1, v_3), (v_3, v_2), and (v_2, v_1); and region r_3 by edges (v_1, v_2), (v_2, v_3), (v_3, v_4), and (v_4, v_1). An embedding of a graph G on a surface S is a 2-cell embedding if all embedded regions are 2-cells. A region is a 2-cell if any simple closed curve within the region can be collapsed to a single point. Figure 3(b) shows that region r_1 is a 2-cell. The last term we need to define is girth. The girth of a graph G, denoted girth (G), is the length of the shortest cycle in graph G.



a. Graph with three regions.



b. Region r_1 is two-cells.

Figure 3.

Theorem 2.1 (General Euler's Formula):
Let $G = (V, E)$ be a connected graph with a 2-cell embedding on a surface of genus γ and having r regions. Then

$$|V| - |E| + r = 2 - 2\gamma \tag{1}$$

The proof may be found in [4] and will be omitted here. The next theorem gives a lower bound on the genus γ of a graph G.

Theorem 2.2: If $G = (V, E)$ is a connected graph with girth (G) = α then

$$\gamma(G) \geq \frac{|E|\left(1 - \frac{2}{\alpha}\right) - |V|}{2} + 1 \tag{2}$$

Proof: The following proof can be found in [4] and is paraphrased here. Since α = girth (G) is the length of the shortest cycle in graph G, the boundary of every region contains at most α edges, and since every edge is on the boundary of at least two regions, $\alpha r \leq 2|E|$. The result follows by substituting the value for r into Theorem 2.1 and simplifying.

3. Girth of Pancake Graph

Claim: Let $P_n = (S_n, E_n)$ be a pancake graph of dimension $n > 2$. Then

$$\text{girth}(P_n) = 6 \tag{3}$$

Proof: It is clear that the claim is true for $n = 3$. It is also clear that, by induction, the shortest cycle in each separate copy of P_{n-1} in P_n has the length of 6. Let consider the cycle that involves two or more copies of P_{n-1}. As mentioned in the introduction, vertices in each copy of P_{n-1} share the same ending element. If the cycle starts from permutation $[p_1, p_2, \ldots, p_i, p_{i+1}, \ldots, p_n]$ in the P_{n-1,p_n}, this permutation has only one connection (denoted by the external connection) to a permutation in another copy of P_{n-1} (P_{n-1,p_1}) by the n-flip. Hence, the cycle has to pass through at least 2 vertices (permutations) in each copy of P_{n-1}. Imagine that the 2 vertices of P_{n-1,p_n} is connected by the i-flip, the external connections of these 2 vertices do not lead to the same copy of P_{n-1} Figure 4. This means that the cycle has to involve at least 3 different copies of P_{n-1}, each copy in turn involves at least 2 vertices; or that the cycle has to involve at least 2 distinct copies of P_{n-1}, each copy in turn involves at least 3 vertices. Both cases yield the length of 6 or more for the cycle.

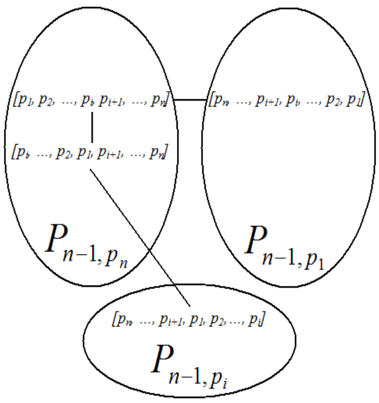


Figure 4. Cycle involve 3 different copies of P_{n-1}.

4. Genus of Pancake Graph

Claim 1: The lower bound for the genus of pancake graph of dimension n is

$$n!\left(\frac{n-4}{6}\right)+1 \tag{4}$$

Proof: For $P_n = (S_n, E_n)$, we have $|S_n| = n!$, $|E_n| = \frac{n!(n-1)}{2}$ and girth(P_n) = 6. By substituting these values into Theorem 2.2 and simplifying, we have

$$\gamma(P_n) \geq n!\left(\frac{n-4}{6}\right)+1 \tag{5}$$

Claim 2: The upper bound for the genus of pancake graph of dimension n is

$$n!\left(\frac{n-3}{4}\right)-\frac{n}{2}+1 \tag{6}$$

Proof: From the General Euler's Formula $|V|-|E|+r = 2-2\gamma$, we have

$$\gamma = \frac{|E|-|V|-r}{2}+1 \tag{7}$$

The upper bound of the genus is obtained when the number of regions is minimum. Since the pancake graph P_n of dimension n has n copies of P_{n-1}, the minimum number of regions in P_n is also n. Substituting all values of |E|, |V| and the minimum number of regions r into the formula above and then simplifying, we obtain:

$$\gamma(P_n) \leq n!\left(\frac{n-3}{4}\right)-\frac{n}{2}+1 \tag{8}$$

Combining claims 1 and 2, we obtain:

$$n!\left(\frac{n-4}{6}\right)+1 \leq \gamma(P_n) \leq n!\left(\frac{n-3}{4}\right)-\frac{n}{2}+1 \tag{9}$$

5. Comparison of the Genuses of the Pancake, the Star, and Hypercube

The pancake graph P_n has the same number of nodes and edges with the star graph G_n. However, according to the result we got in the previous section, the lower bound of the genus of the pancake graph is equal the genus of the star graph, while its upper bound is approximately one and a half as much. We compare the pancake network with its equivalent size hypercube and star networks. Hoelzeman and Bettayeb compared the genus of star graph and hypercube in [11]. We are now expanding that comparison by adding the upper bound of the pancake graph to the table and considering the genus of star graph as the lower bound for the genus of pancake graph.

Table 1. Comparison of P_n, G_n and $Q_{\lg(n!)}$.

n	lg(n!)	$\gamma(G_n)$	$\gamma(P_n)$	$\gamma(Q_{\lg(n!)})$
2	1	0	0	0
3	3	0	0	0
4	4	1	5	1
5	7	21	58	49
6	9	241	538	321
7	12	2521	5037	4097
8	15	26881	50397	45057
9	18	302401	544316	458753
10	22	3628801	6350396	9437185

Table 1 indicates that the upper bound of the genus of the pancake graph is still less than the genus of the hypercube. We prove that this is true by directly comparing them. The genus of the binary hypercube of dimension m is $\gamma(Q_m) = (m-4)2^{m-3}+1$ [3]. Thus, $\gamma(P_n) < \gamma(Q_{\lg(n!)})$, if, and only if,

$$n!\left(\frac{n-3}{4}\right)-\frac{n}{2}+1 < n!\left(\frac{\lg(n!)-4}{8}\right)+1 \tag{10}$$

The left side is less than $n!\left(\frac{n-3}{4}\right)+1$ (11)

and $$n!\left(\frac{n-3}{4}\right)+1 < n!\left(\frac{lg(n!)-4}{8}\right)+1 \quad (12)$$

$$\Leftrightarrow \ 2(n-3) < lg(n!)-4 \quad (13)$$

This inequality is satisfied for all n > 6 since n grows less rapidly than lg(n!) does. Hence, we have

$$\gamma(P_n) < \gamma(Q_{\lg(n!)}) \qquad \forall n > 6. \quad (14)$$

6. Conclusions

The pancake graph has the same number of nodes and edges as the star graph, but the diameter of the pancake graph is smaller. However, the genus of the pancake graph is not as good as the genus of the star graph. That is, its lower bound is equal the genus of the star graph and its upper bound is approximately one and a half times as much as that of the star graph. In comparison with the binary hypercube, both the star graph and the pancake graph have a much smaller vertex degree and diameter. Its smaller genus also indicates that the pancake graph has a more efficient layout than the hypercube. The genus of the pancake graph is bounded as follows:

$$n!\left(\frac{n-4}{6}\right)+1 \le \gamma(P_n) \le n!\left(\frac{n-3}{4}\right)-\frac{n}{2}+1 \quad (15)$$

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